LAMINAR FREE-CONVECTION ON A VERTICAL FLAT PLATE WITH UNIFORM SURFACE TEMPERATURE OR UNIFORM SURFACE HEAT FLUX

by 149

JEROME CHIEH-JAN LEE

B. S., National Taiwan University, Taipei, Taiwan, 1950

A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

Approved by:

N. Z. Azer

Major Professor

LD
2668
.T4
1967
L478
c.2

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Natural or free-convection, distinct from forced-convection, is the heat-transfer mechanism observed as a result of the motion of the fluid due to the action of body forces which are imposed on the fluid by the density gradients arising from the temperature difference.

Free-convection flow is a principal mode of heat-transfer in many engineering applications such as in the fields of nuclear engineering, aeronautics, and gas turbines industry. In the last decade, the problem of free-convection flow under the influence of the gravitational field has been investigated extensively---analytically as well as experimentally.

If body forces other than gravitational forces, such as centrifugal or Coriolis forces, exist, approximate solutions can be obtained by replacing the gravitational acceleration in Grashof's number by the acceleration of the body forces in the problem of interest.

Several investigations on the subject of this report were published by Schmidt and Beckmann (12),* Saunders (11), Schuh (13), Squire (5), Ostrach (9), Sparrow (14), Sparrow and Gregg (15), Eckert (2), and others. Solutions dealing with different geometries, various boundary conditions, and the effects of property variations with temperature were found in literature. In general, there are two approaches to the solution of heat flow by free-convection, namely, exact solutions and approximate solutions of the

*Numbers in parenthesis refer to references in bibliography.

boundary-layer equations.

Sevaral analytical investigations using the approximate integral method have been reported by Squire (5), Eckert (2), Sparrow (14), and others. All these analytical approaches have been based on the assumption that the thicknesses of the thermal and the velocity boundary-layers are very nearly equal. However, results reported by Ostrach (9) for the analysis of laminar free-convection flow and heat transfer about a flat vertical plate at constant wall temperature, showed that the thicknesses of the thermal and the velocity boundary-layera are different and the relative thickness depends on Prandtl number. For fluids whose Prandtl numbers are greater than one, the thickness of the velocity boundary-layer is greater than that of the thermal boundary-layer, while for fluids whose Prandtl numbers are less than one, the thickness of the thermal boundary-layer is greater than that of the velocity boundary-layer.

The object of this report is to analyze the gravitational free-convection problem around a vertical plate under various boundary conditions for the cases of $Pr \geq 1$ and $Pr \leq 1$* and to present some physical interpretationa of the heat transfer mechanism of free-convection, since this mechanism is of vital importance in various engineering applications. In the analytical treatment of the problem, the technique of the approximate integral method, without assuming $\delta_h = \delta_t$, will be used.

In this analysis, the natural flow existing around the vertical plate is assumed to display a laminar boundary-layer structure so that the Grashof number is $10^9 > Gr > 10^4$ (8). The thermal boundary-layer thickness is defined here as the distance perpendicular to the plate surface at which the

---

*See nomenclature.

difference between the fluid temperature in the immediate neighborhood of the plate surface and the ambient fluid temperature and the derivative of the ambient fluid temperature with respect to y cease to be perceptible. Similarly, the velocity boundary-layer thickness is defined as the distance perpendicular to the plate surface where the tangential velocity u and its derivative with respect to y cease to be perceptible.

In accord with the usual practice in free-convection analyses, the density will be considered a variable only in the buoyancy term. All other properties will be assumed constant. Viscous dissipation and work against the gravity field will be neglected.

CHAPTER 2

LITERATURE SURVEY

The problem of laminar free-convection heat transfer from a heated vertical plate in still air was first considered by Lorenz in 1881 (6), by assuming that the temperature and velocity at any point of the flow field depended only on the distance from the plate. Since then, numerous theoretical investigations have been conducted on the problem under consideration---either by using the exact solution method, that is, the complete solutions of the boundary-layer equations for given set of boundary conditions, or by approximate solution method which is usually obtained from the solutions of the integral momentum and energy equations.

In the following sections, a literature survey for the above-mentioned methods will be presented.

2.1. EXACT SOLUTIONS

2.1.1. Flat Plate with Uniform Surface Temperature.

The first exact solution of the free-convection problem for the vertical flat plate was developed by Pohlhausen in 1921 (10). The basic boundary-layer equations with constant properties were considered. By the introduction of the stream function and a suitable similarity transformation, he demonstrated that the partial differential equations could be reduced to ordinary differential equations. Integration of the resulting equations was done for air (Prandtl number = 0.733).

Schmidt and Beckmann (12) conducted excellent experimental and

theoretical studies on the free-convection flow of air subjected to the gravitational force about a vertical flat plate. Their experiments showed that Lorenz' assumption was invalid. In their theoretical development, they assumed that the thicknesses of the layers in which the temperature and the velocity differed appreciably from the values at infinity were small compared with the height of the plate, and the assumptions were verified by experimental observation. Their experimental measurements of the temperature profiles agreed well with their analytical results. Eckert (3) further verified the experimental results of Schmidt and Beckmann (12) by means of Zehnder-Mach interferometer studies. Schuh (13) extended the numerical calculations of Pohlhausen's method by integrating the same equations for Prandtl numbers of 0.733, 10, 100, and 1000. His calculated velocity and temperature distributions showed that, as Prandtl number increased above unity, the viscous boundary-layer became progressively thicker than the thermal boundary-layer.

Ostrach (9) performed further calculations using electronic computer techniques to obtain numerical solutions of the free-convection boundary-layer equations for Prandtl numbers of 0.01, 0.72, 0.733, 1, 2, 10, 100, and 1000. These values of Prandtl number are representative of liquid metals, gases, liquids, and very viscous fluids. It was found that the Grashof number was the principal factor which determined the type of flow and, for large Grashof numbers ($Gr > 10^4$), the flow was of the boundary-layer type. Computation results for air of $Pr = 0.72$ and $Pr = 0.733$ were presented in Ostrach's paper. Ostrach's results for $Pr = 0.733$ were compared with the previously obtained results of Schmidt and Beckmann (12), and they were found to be in agreement. Results for the heat-transfer coefficients were in agreement with other experimental and approximate theoretical

investigations over the range of Prandtl numbers covered by both investigations. Ostrach noted that the velocity and thermal boundary-layer thicknesses could be estimated from the dimensionless velocity and temperature profiles presented in his paper and pointed out that, for Pr >> 1, the velocity boundary-layer was much thicker than the thermal boundary-layer. However, no conclusion was drawn regarding the two boundary-layers thicknesses for low Prandtl numbers, though his results showed a noticeable difference between their thicknesses for Pr = 0.01.

2.1.2. Flat Plate with Uniform Surface Heat Flux.

Sparrow and Gregg (15) analyzed the problem of laminar free-convection from a vertical plate with uniform surface heat flux, by integrating the transformed momentum and energy equations. The modified Grashof number, $Gr_x^* \ (= \frac{g\beta q x^4}{k \nu^2})$, was introduced in his analysis to replace the conventional Grashof number, $Gr_x \ (= \frac{g\beta(T_w - T_\infty)x^3}{\nu^2})$, because the former contains quantities which would all be known at the beginning of a calculation whereas the latter contains a temperature difference which is unknown at the beginning of a calculation. Surface temperature variation and local Nusselt numbers were calculated for Prandtl numbers 0.1, 1, 10, and 100. Results of the Numerical calculations of $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ versus Pr were extrapolated to a Prandtl number of 0.01. Heat-transfer results were expressed in terms of an average Nusselt number which was based on the difference between the average wall temperature and the ambient fluid temperature. A comparison between their results and those of Ostrach (9) was made for the range of Prandtl numbers from 0.1 to 100. The resulting prediction of the heat transfer coefficient for constant wall heat flux was found to be always

higher than the constant wall temperature results for the same local temperature difference $(T_w - T_\infty)_x$. However, the difference was less than 8%. Further, it was pointed out that the experimentally determined Nusselt numbers based on the temperature difference halfway along the plate were very close to those for the constant temperature case.

2.2. APPROXIMATE SOLUTIONS

2.2.1. Flat Plate with Uniform Surface Temperature.

(i) Integral Method (or Von Karman-Pohlhausen Method).

The integral method most often employed in free-convection boundary-layer flows is attributed to Squire (5). He analyzed the problem under consideration by assuming a polynomial for the velocity and temperature profiles which could be made to satisfy the boundary conditions. The temperature distribution assumed for air was in fair agreement with the values calculated by Schmidt and Beckmann (12), while the assumed velocity distribution was not.

Eckert et al. (2) also analyzed the same problem by introducing the same simplified assumptions of the boundary-layer as Squire did. For air, the heat-transfer result calculated by his approximate solution agreed quite well with the exact analytic solution obtained by Pohlhausen and the experimental results of Schmidt and Beckmann. Nusselt number results were also in good agreement with the results of Ostrach (9): within 10% in the range of Prandtl numbers from 0.01 to 1000. Eckert, also, pointed out the fact that, for high Prandtl numbers, the velocity boundary-layer is expected to be thicker than the thermal boundary-layer. Furthermore, Eckert mentioned that the inner part of the velocity profile, between the wall and the point of maximum velocity, might change very little with Prandtl number in relation

to the temperature profile, but that the outer part of the velocity profile from the location of maximum velocity to the edge of the boundary-layer, might become thicker, because the driving force for the flow results from the temperature differences.

The assumptions made in the Squire-Eckert analyses were as follows:

1. The thicknesses of the boundary-layers are finite.
2. The difference between the thermal and the velocity boundary-layers thicknesses is negligible, namely, $\delta_h = \delta_t$.
3. The expressions for the velocity and temperature profiles are given by

$$u = u_x \frac{y}{\delta} (1 - \frac{y}{\delta})^2$$

$$T - T_\infty = (T_w - T_\infty)(1 - \frac{y}{\delta})^2$$

where u_x is a characteristic velocity which is a function of x and is to be determined. Their approximate methods resulted in a relation between the local Nusselt number, Grashof number and Prandtl number of the form

$$Nu_x = 0.508 \, Pr^{\frac{1}{2}} (Pr + \frac{20}{21})^{-\frac{1}{4}} (Gr_x)^{\frac{1}{4}}$$

Yamagata (16) made assumptions similar to those in Squire-Eckert analysis. However, he tried to make the velocity and temperature profiles more general than those used in Squire's analysis. A free parameter which depends on Prandtl number was introduced in the velocity profile in order that the form of the profile could change with Prandtl number over the whole range of the practical interest. The following expressions were assumed for the boundary-layer profiles:

$$u = u_x \phi(\eta), \quad \phi(\eta) = \eta(1 - \eta)^3 \{1 + (3 - \lambda)\eta\}$$

$$T = T_\infty + (T_w - T_\infty)\theta(\eta), \quad \theta(\eta) = (1-\eta)^3\,(1+\eta)$$

where $\eta = \frac{y}{\delta}$, and δ is the thickness of the boundary-layer. λ is a parameter to be determined and is a function of Prandtl number. Also, the following boundary-condition was introduced:

$$\left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0} = -\frac{g\beta(T_w - T_\infty)}{\nu}$$

It was noted that the velocity within the free-convection layer increased gradually with the distance from the wall and, after the maximum value was reached, velocity decreased asymptotically to zero. Yamagata pointed out that it is very difficult to define the thickness of the velocity boundary-layer in free-convection for Prandtl number smaller than unity, because the asymptotic decrease of the velocity profile to zero value made the thickness of the boundary-layer ambiguous. He assumed the following general expression for the Nusselt number,

$$Nu_x = K_\varepsilon\,(Gr_x \cdot Pr^\varepsilon)^{\frac{1}{4}}$$

where ε is another parameter introduced. For the limiting cases of zero and infinite Prandtl numbers, the Nusselt number becomes

$$Nu_x = K_1\,(Gr_x \cdot Pr)^{\frac{1}{4}} \quad \text{for } Pr \to \infty$$

$$Nu_x = K_2\,(Gr_x \cdot Pr^2)^{\frac{1}{4}} \quad \text{for } Pr \to 0$$

In other words, the parameter ε varies in the range of $1 < \varepsilon < 2$ for Prandtl number in the range of $\infty > Pr > 0$. Values of ε corresponding to each Prandtl number were calculated in his paper.

Fujii (4) proposed a "Modified Integral Method" to supplement inaccurate

results of the velocity distribution obtained from Squire's method. He suggested a velocity profile which included the feature of variable thickness and had a free parameter, s, which depends on Prandtl number. Two approximate solutions were presented, and the velocity and temperature profiles were assumed respectively as follows;

1st method: For $Pr < 0.01$

$$\phi(s\eta) = s\eta e^{-s}$$

$$\theta(\eta) = (1 + \tfrac{1}{2}\eta)e^{-\eta}$$

2nd method: For $0.01 < Pr < 1000$

$$0 \leq \eta < \infty: \quad \phi(s\eta) = s\eta e^{-s\eta}$$

$$0 \leq \eta < 1: \quad \theta(\eta) = (1 + \eta)\,(1 - \eta)^3$$

$$1 \leq \eta \leq \infty: \quad \theta(\eta) = 0$$

where $\eta = y/\delta$, $u = u_x\phi(s\eta)$, $\theta = \theta(\eta)$.

It waa pointed out that the second method would be most suitable as a supplement to the Squire's solution (5), while the first solution would be the simplest and the most accurate for liquid metals. The calculated results showed satisfactory agreement with those of Ostrach (9) and Squire (5).

(ii) Meksyn's Method.

Meksyn (7) devised an analytic technique for solving boundary layer equations. It was reported in a series of papers which appeared in the Proceedings of the Royal Society of London beginning 1948. Brindley (1)

applied this technique to the classical problem of a free-convection flow at a constant wall temperature for a vertical wall placed in an infinite fluid at rest. He found that the first three terms of the asymptotic series would provide a good approximation. The computations were based on the first three terms of the appropriate gamma function expansion for the temperature and stream functions. Numerical results obtained for Pr = 0.733 were compared with those of Schmidt and Beckmann (12), and Squire (5). Values of the mean Nusselt numbers agreed within 4% with those of Ostrach (9) for various values of Pr $\geqq$ 1. However, for liquid metal, the technique did not give good results due to mathematical difficulties.

2.2.2. Flat Plate with Prescribed Nonuniform Surface Heat Flux or Prescribed Nonuniform Surface Temperature.

Sparrow (14) analyzed the problem of flat plate with prescribed nonuniform surface heat flux or prescribed nonuniformed surface temperature by the integral method. He assumed that the thicknesses of the velocity and the temperature boundary-layers were very nearly equal. The two boundary-layers profiles were approximated by the following polynomials:

Velocity profile: $$u = u_x \left(\frac{y}{\delta}\right)\left(1 - \frac{y}{\delta}\right)^2$$

Temperature profile:

for prescribed nonuniform wall heat flux: $$T - T_\infty = \frac{q\delta}{2k}\left(1 - \frac{y}{\delta}\right)^2$$

for prescribed nonuniform wall temperature: $$T - T_\infty = (T_w - T_\infty)\left(1 - \frac{y}{\delta}\right)^2$$

where u_x is the characteristic velocity and δ is the boundary-layer thickness, and both are functions of x and Pr.

Results were presented (1) for the wall-temperature distribution at a prescribed distribution of wall heat flux, and (2) for the wall heat flux along the plate at any prescribed distribution of wall temperature. Local heat-transfer coefficients for both cases could be obtained from Sparrow's results. The numerical calculations were made for fluids having Prandtl numbers in the range of 0.01 to 1000. However, the derived results of Sparrow's analysis could not be checked due to the unavailability of any corresponding experimental data or any similar analysis for range of Prandtl numbers considered. For the special cases of uniform wall temperature and uniform heat flux, the heat-transfer results obtained were in good agreement with those of other exact solutions and available experimental measurements.

CHAPTER 3

ANALYSIS

3.1. BASIC EQUATIONS

The physical model and the co-ordinate system to be used are shown in Fig. 1. The x-direction extends vertically upward from the lower edge of the plate for the case of heat transfer from the plate to the fluid and downward from the upper edge of the plate for the case of heat transfer from the fluid to the plate. The y-direction is measured in the direction of the outward normal to the plate.

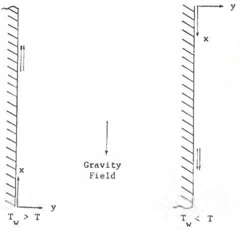
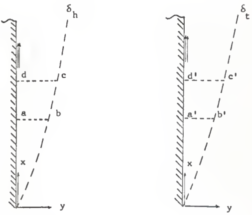


Fig. 1. Co-ordinate Systems

The analysis will be performed for the case of heat transfer from the wall to the fluid, but the results apply to both cases. A flat plate heated to a temperature T_w is suspended in a large body of fluid, which is at temperature T_∞. In the neighborhood of the heated plate, the fluid rises because of the buoyancy force.

Integral Momentum Equation

To derive the equations governing the motion of the fluid, we consider the momentum balance and the energy balance for the aggregate of fluid particles within the control volume, abcd, in the boundary-layer of Fig. 2(a).



(a) (b)

Fig. 2. Control Volume in Boundary-layer

Let us consider a unit length in the direction normal to the x-y plan.

The rate of x-momentum inflow through face ab is

$$\rho \int_0^{\delta_h} u^2 dy$$

and the rate of x-momentum outflow through face cd is

$$\rho \int_0^{\delta_h} u^2 dy + \rho \frac{d}{dx} \left(\int_0^{\delta_h} u^2 dy \right) dx$$

No momentum crosses face bc, because x-component of the velocity at the edge of the velocity boundary-layer is zero. The net outflow rate of x-component momentum is therefore

$$\rho \frac{d}{dx} \left(\int_0^{\delta_h} u^2 dy \right) dx$$

The net outflow flow rate of x-component momentum is equal to the summation of the forces in the x-direction acting on the surfaces of the control volume. These forces in free-convection are

1. The shearing stress at the surface ad, $-(\tau_{yx})_{y=0}\ dx$
2. The pressure on face ab, $\int_0^{\delta_h} p dy$
3. The pressure on face cd, $-\left\{ \int_0^{\delta_h} p dy + \frac{d}{dx} \left(\int_0^{\delta_h} p dy \right) dx \right\}$
4. The gravitational forces acting on the control volume, $-\int_0^{\delta_h} \rho g dx dy$

Since the velocity gradient, $\frac{\partial u}{\partial y}$, on face, bc, is zero, no shearing stress exists on the face, bc.

Equating the forces to the rate of outflow of x-component momentum

yields

$$\rho \frac{d}{dx} \int_0^{\delta_h} u^2 dy = -(\tau_{yx})_{y=0} - \delta_h \left(\frac{dp}{dx}\right) - \int_0^{\delta_h} \rho g \, dy$$

However,

$$(\tau_{yx})_{y=0} = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$\frac{dp}{dx} = -\rho_\infty g$$

$$\frac{\rho - \rho_\infty}{\rho} = -\beta (T - T_\infty)$$

where β is the coefficient of volumetric expansion of the fluid and ρ_∞ is the density of the stagnant fluid. Hence,

$$\rho \frac{d}{dx} \int_0^{\delta_h} u^2 dy = -\mu \left(\frac{\partial u}{\partial y}\right)_{y=0} + \delta_h \rho_\infty g - \int_0^{\delta_h} \rho g \, dy$$

$$= -\mu \left(\frac{\partial u}{\partial y}\right)_{y=0} - \int_0^{\delta_h} (\rho - \rho_\infty) g \, dy$$

$$\frac{d}{dx} \int_0^{\delta_h} u^2 dy = -\nu \left(\frac{\partial u}{\partial y}\right)_{y=0} + g \beta \int_0^{\delta_h} (T - T_\infty) \, dy \tag{3.1.1}$$

Integral Energy Equation

We consider the control volume, a'b'c'd', of Fig. 2(b).

Energy is convected into and out of the control volume, a'b'c'd', as a result of the fluid motion, and there is also heat flow by conduction across the interface. The energy flow rates across the individual faces of the control volume, a'b'c'd', are as follows:

1. Rate of energy convected into the control volume through face, a'b',

$$\rho c_p \int_0^{\delta_t} uT\,dy$$

2. Rate of energy convected into the control volume through face, b'c',

$$c_p T_\infty \frac{d}{dx}\left(\int_0^{\delta_t} \rho u\,dy\right)dx$$

3. Rate of energy convected out of the control volume through face, c'd',

$$\rho c_p \int_0^{\delta_t} uT\,dy + \rho c_p \frac{d}{dx}\left(\int_0^{\delta_t} uT\,dy\right)dx$$

4. Rate of heat conducted through face, a'd',

$$-k\left(\frac{\partial T}{\partial y}\right)_{y=0} dx$$

From the first law of thermodynamics, one can write

$$\frac{d}{dx}\int_0^{\delta_t} (T - T_\infty)\,u\,dy = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{3.1.2}$$

Equations (3.1.1) and (3.1.2) are the basic integral momentum and energy equations. They can also be derived by integrating the following three equations which express conservation of mass, momentum, and energy for steady laminar flow in the boundary-layers.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.1.3}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta(T - T_\infty) \tag{3.1.4}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{3.1.5}$$

The limits of integration in such a case will be from $y = 0$ to $y = \delta$.

3.2. NEW ANALYSIS

In this analysis the following four particular problems will be considered:

I. Flat plate with uniform surface temperature.

1. The case of high Prandtl number, $Pr > 1$.
2. The case of low Prandtl number, $Pr < 1$.

II. Flat plate with uniform surface heat flux.

3. The case of high Prandtl number, $Pr > 1$.
4. The case of low Prandtl number, $Pr < 1$.

As mentioned in the literature survey, these problems have been extensively studied, both theoretically and experimentally. The selection of the above mentioned cases will provide a basis for checking the validity of the analysis presented in this report.

In spite of the fact that the integral method has been used in analyses of free-convection boundary-layer flow problems to predict heat transfer with acceptable accuracy, error in skin friction calculated by this method could be significant. In addition, the theoretical relations derived by the previous investigations predict much lower results for heat transfer to molten metals ($Pr << 1$) than those results obtained from the exact solutions. These discrepancies are the result of the choice of temperature and velocity profiles and the assumption that the velocity and thermal boundary-layers are identical in thickness. Thus, it is possible that the degree of

accuracy of the heat transfer prediction by the integral method, especially at low Prandtl number, could be improved by assuming $\delta_h \neq \delta_t$ as shown in Fig. 3.

For simplicity, we introduce the ratio of the thicknesses of the boundary-layers as follows: $\zeta = \frac{\delta_t}{\delta_h}$ for fluids of $Pr > 1$, $\zeta < 1$, and $\xi = \frac{\delta_h}{\delta_t}$ for fluids of $Pr < 1$, $\xi < 1$.

The integrals in the integral momentum and energy equations will be evaluated by dividing the boundary-layers into two parts; namely, for $Pr > 1$, $y = 0$ to $y = \delta_t$ and from $y = \delta_t$ to $y = \delta_h$, and for $Pr < 1$, $y = 0$ to $y = \delta_h$ and from $y = \delta_h$ to $y = \delta_t$.

3.2.1. Flat Plate with Uniform Surface Temperature.

The velocity and temperature profiles in the neighborhood of the plate with uniform surface temperature are shown in Fig. 3 for the cases of $Pr > 1$ and $Pr < 1$, and in this analysis they will be approximated by the following polynomials:

$$u = u_x \left(\frac{y}{\delta_h}\right)\left(1 - \frac{y}{\delta_h}\right)^2 \tag{3.2.1.1}$$

$$T - T_\infty = (T_w - T_\infty)\left(1 - \frac{y}{\delta_t}\right) \tag{3.2.1.2}$$

where the characteristic velocity u_x and the boundary-layer thicknesses δ_h and δ_t are functions of x and Pr and remain to be determined. Equations (3.2.1.1) and (3.2.1.2) satisfy the following boundary conditions:

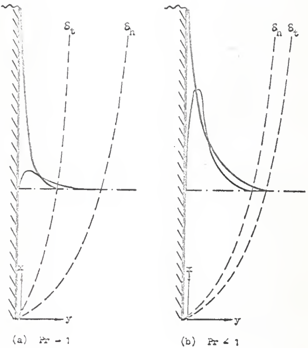


Fig. 3. Laminar Free-Convection Heat Transfer from a Vertical Flat Plate Surface to Fluid

$$\text{At } y = 0; \quad u = 0, \quad \text{and} \quad T = T_w \tag{3.2.1.3}$$

$$\text{At } y = \delta_h; \quad u = 0, \quad \text{and} \quad \frac{\partial u}{\partial y} = 0 \tag{3.2.1.4}$$

$$\text{At } y = \delta_t; \quad T = T_\infty, \quad \text{and} \quad \frac{\partial T}{\partial y} = 0 \tag{3.2.1.5}$$

Case 1. High Prandtl Number, $Pr > 1$.

For fluids of $Pr > 1$ the kinematic viscosity is greater than the thermal diffusivity; as a result, the viscous effects penetrate much deeper into the fluid than the thermal effects. Therefore, as illustrated in Fig. 3(a), the velocity boundary-layer can be assumed to be thicker than the thermal boundary-layer. It is also to be noted that despite the fact that there is no temperature gradient in the region between $y = \delta_t$ and $y = \delta_h$, the fluid is still in motion. This is due to the drag action created by the motion of the fluid near the edge of the thermal boundary layer.

Substituting Equations (3.2.1.1) and (3.2.1.2) into the integral momentum Equation (3.1.1),

$$\frac{1}{105}\frac{d}{dx}(\delta_h \cdot u_x^2) = \frac{1}{3} g \beta (T_w - T_\infty) \xi \delta_h - \nu \frac{u_x}{\delta_h} . \tag{3.2.1.6}$$ *

Similarly, substituting the assumed velocity and temperature profiles into the integral energy Equation (3.1.2) yields

$$\frac{1}{6}\frac{d}{dx}(M \xi \delta_t u_x) = \frac{\alpha}{\delta_t} , \tag{3.2.1.7}$$ *

where

---

*The details of the derivation of Equations (3.2.1.6) and (3.2.1.7) are given in Appendices 1.1. and 1.2. respectively.

$$M = \frac{1}{4} - \frac{1}{5}\zeta + \frac{1}{20}\zeta^2$$

and $$\zeta = \frac{\delta_t}{\delta_h}$$

In order to solve the differential Equations (3.2.1.6) and (3.2.1.7), tha assumptions are made that both u_x and δ_h follow a power-law variation with x and that the ratio ζ is not a function of x. Thus,

$$u_x = C_{11}\, x^{m_1} \tag{3.2.1.8}$$

and $$\delta_h = C_{12}\, x^{n_1} . \tag{3.2.1.9}$$

Introducing Equations (3.2.1.8) and (3.2.1.9) into Equations (3.2.1.6) and (3.2.1.7) gives

$$\frac{2m_1+n_1}{105} C_{11}{}^2 C_{12}\, x^{2m_1+n_1-1} = \frac{1}{3} g\beta\,(T_w - T_\infty)\, C_{12}\, x^{n_1}\zeta - \frac{C_{11}}{C_{12}}\,\nu\, x^{m_1-n_1} \tag{3.2.1.10}$$

and

$$\frac{m_1+n_1}{6} C_{11}C_{12}{}^2\, M\zeta^3\, x^{m_1+2n_1-1} = \alpha . \tag{3.2.1.11}$$

Since Equations (3.2.1.10) and (3.2.1.11) must be valid for any value x, the value of the exponent of x must be the same on both sides of each equation so that

$$2m_1+n_1-1 = n_1 = m_1-n_1$$

and $$m_1+2n_1-1 = 0 ,$$

from which

$$m_1 = \frac{1}{2} \text{ and } n_1 = \frac{1}{4} .$$

Substituting these values of m_1 and n_1 back into Equations (3.2.1.10) and (3.2.1.11) yields

$$C_{11}{}^2 C_{12} = 84 \left\{ \frac{1}{3} g \beta (T_w - T_\infty) C_{12} \zeta - \frac{C_{11}}{C_{12}} \nu \right\} \qquad (3.2.1.12)$$

and

$$C_{11} C_{12}{}^2 = 8 \alpha (M \zeta^3)^{-1} . \qquad (3.2.1.13)$$

Solving Equations (3.2.1.12) and (3.2.1.13) simultaneously for C_{11} and C_{12}, we get

$$C_{11} = (\tfrac{8}{3})^{\frac{1}{2}} M^{-\frac{1}{2}} \zeta^{-1} \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{1}{2}} \left\{ \frac{g \beta (T_w - T_\infty)}{\nu 2} \right\}^{\frac{1}{2}} \nu \qquad (3.2.1.14)^*$$

and

$$C_{12} = (24)^{\frac{1}{4}} M^{-\frac{1}{4}} \zeta^{-1} \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{\frac{1}{4}} \left\{ \frac{g \beta (T_w - T_\infty)}{\nu 2} \right\}^{-\frac{1}{4}} (Pr)^{-\frac{1}{2}} \qquad (3.2.1.15)^*$$

The resultant expressions for u_x, δ_h and δ_t are

$$u_x = (\tfrac{8}{3})^{\frac{1}{2}} M^{-\frac{1}{2}} \zeta^{-1} \nu \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{1}{2}} \left\{ \frac{g \beta (T_w - T_\infty)}{\nu 2} \right\}^{\frac{1}{2}} x^{\frac{1}{2}} , \qquad (3.2.1.16)$$

---

*The details of the derivation of Equations (3.2.1.14) and (3.2.1.15) are given in Appendix I.3.

$$\frac{\delta_h}{x} = (24)^{\frac{1}{4}} M^{-\frac{1}{4}} \zeta^{-1} \left\{\frac{2}{21} \zeta^{-3} M^{-1} + Pr\right\}^{\frac{1}{4}} Gr_x^{-\frac{1}{4}} Pr^{-\frac{1}{2}} , \tag{3.2.1.17}$$

and

$$\frac{\delta_t}{x} = (24)^{\frac{1}{4}} M^{-\frac{1}{4}} \left\{\frac{2}{21} \zeta^{-3} M^{-1} + Pr\right\}^{\frac{1}{4}} Gr_x^{-\frac{1}{4}} Pr^{-\frac{1}{2}} . \tag{3.2.1.18}$$

The rate of heat flow from the plate surface is given by the following equation

$$q = - k\left(\frac{\partial T}{\partial y}\right)_{y=0} = h_x(T_w - T_\infty) \tag{3.2.1.19}$$

where h_x is the local film coefficient of heat transfer.

Using the temperature distribution of Equation (3.2.1.2), we obtain

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = - \frac{2}{\delta_t} (T_w - T_\infty) . \tag{3.2.1.20}$$

Therefore,

$$h_x = \frac{2k}{\delta_t} . \tag{3.2.1.21}$$

By definition, the local Nusselt number is

$$Nu_x = \frac{h_x x}{k} . \tag{3.2.1.22}$$

Substituting Equations (3.2.1.18) and (3.2.1.21) into Equation (3.2.1.22) gives

$$Nu_x = \frac{2x}{\delta_t}$$

$$= 2(24)^{-\frac{1}{4}} M^{\frac{1}{4}} \left\{\frac{2}{21} \zeta^{3} M^{-1} + Pr\right\}^{-\frac{1}{4}} Gr_x^{\frac{1}{4}} Pr^{\frac{1}{2}}$$

or

$$Nu_x = (\tfrac{2}{3})^{\frac{1}{4}} M^{\frac{1}{4}} \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{1}{4}} Gr_x^{\frac{1}{4}} Pr^{\frac{1}{2}} . \qquad (3.2.1.23)$$

By rearranging Equation (3.2.1.23), it follows that

$$Nu_x = (\tfrac{2}{3})^{\frac{1}{4}} \left\{ \frac{M\ Pr^2}{\frac{2}{21} \zeta^{-3} M^{-1} + Pr} \right\}^{\frac{1}{4}} Gr_x^{\frac{1}{4}} \qquad (3.2.1.24)$$

This equation can also be rewritten in the form

$$\frac{Nu_x}{\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}} = (\tfrac{8}{3})^{\frac{1}{4}} \left\{ \frac{M\ Pr^2}{\frac{2}{21} \zeta^{-3} M^{-1} + Pr} \right\}^{\frac{1}{4}} . \qquad (3.2.1.25)$$

Equation (3.2.1.25) will be discussed later.

<u>Case 2</u>. Low Prandtl Number, $Pr < 1$.

For fluids of Prandtl number less than one, the velocity and temperature profiles are qualitatively shown in Fig. 3(b). Because the Prandtl number is less than one, the kinematic viscosity is smaller than the thermal diffusivity. Therefore, the thickness of the velocity boundary-layer can be considered to be smaller than that of the thermal boundary-layer.

Let T_s be the temperature at the edge of the velocity boundary-layer; namely,

$$T_s = (T)_{y=\delta_h}$$

Using T_s, we can write the integral momentum equation as

$$\frac{d}{dx} \int_0^{\delta_h} u^2\, dy = - \nu \left(\frac{\partial u}{\partial y}\right)_{y=0} + g\beta \int_0^{\delta_h} (T - T_s)\, dy \qquad (3.2.1.26)$$

The limits to be applied to the right-hand side of the integral energy equation (3.1.2) can be divided into two parts as shown below:

$$\frac{d}{dx}\int_0^{\delta_t} (T - T_\infty)\, u\, dy = \frac{d}{dx}\int_0^{\delta_h} (T - T_\infty)\, u\, dy + \frac{d}{dx}\int_{\delta_h}^{\delta_t} (T - T_\infty)\, u\, dy$$

$$= -\alpha \left(\frac{\partial T}{\partial y}\right)_{y=0}$$

Since the velocity at the edge of the velocity boundary-layer vanishes, then

$$\int_{\delta_h}^{\delta_t} (T - T_\infty)\, u\, dy = 0$$

As a result, the integral energy equation reduces to

$$\frac{d}{dx}\int_0^{\delta_h} (T - T_\infty)\, u\, dy = -\alpha \left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (3.2.1.27)$$

Substituting Equations (3.2.1.1) and (3.2.1.2) into Equation (3.2.1.26) yields

$$\frac{1}{105}\frac{d}{dx}(\delta_h \cdot u_x^2) = g\beta\,(T_w - T_\infty)\,\xi\,(1 - \frac{2}{3}\xi)\,\delta_h - \nu\frac{u_x}{\delta_h} \qquad (3.2.1.28)^*$$

where ξ is ratio of the velocity boundary layer thickness to the thermal boundary-layer thickness, and this ratio is assumed to be independent of x.

Inserting Equations (3.2.1.1) and (3.2.1.2) into Equation (3.2.1.27) yields

---

*The details of the derivation of Equation (3.2.1.28) are given in Appendix 11.1.

$$\frac{1}{6}\frac{d}{dx}\left(N \cdot \delta_h \cdot u_x\right) = \frac{\sigma}{\delta_t} \qquad (3.2.1.29)^*$$

where

$$N = \frac{1}{4} - \frac{1}{5}\xi + \frac{1}{20}\xi^2 .$$

In order to solve Equations (3.2.1.28) and (3.2.1.29), we introduce assumptions similar to those made in Case 1. Thus,

$$u_x = C_{21}\, x^{m_2} \qquad (3.2.1.30)$$

and

$$\delta_h = C_{22}\, x^{n_2} \qquad (3.2.1.31)$$

Introducing Equations (3.2.1.30) and (3.2.1.31) into Equations (3.2.1.28) and (3.2.1.29) gives

$$\frac{2m_2+n_2}{105} C_{21}^{\ 2} C_{22}\, x^{2m_2+n_2-1} = g\,\beta\,(T_w - T_\infty)\, C_{22}\, x^{n_2}\,(\xi - \frac{2}{3}\xi^2)$$

$$- C_{21}\, C_{22}^{\ -1}\, \nu\, x^{m_2-n_2} \qquad (3.2.1.32)$$

and $$\frac{m_2+n_2}{6} C_{21}\, C_{22}^{\ 2}\, N\, \xi^{-1}\, x^{m_2+2n_2-1} = \alpha . \qquad (3.2.1.33)$$

By the same argument as was employed in Case 1, values of m_2 and n_2 are found to be

$$m_2 = \frac{1}{2}, \quad \text{and} \quad n_2 = \frac{1}{4} .$$

---

*The details of the derivation of Equation (3.2.1.29) are given in Appendix 11.2.

Substituting these values of m_2 and n_2 into Equations (3.2.1.32) and (3.2.1.33) yields

$$C_{21}^{\;2}\, C_{22} = 84 \left\{ g\beta (T_w - T_\infty)\, C_{22}\, (\xi - \tfrac{2}{3}\xi^2) - C_{21}\, C_{22}^{\;-1} \nu \right\} \quad (3.2.1.34)$$

and $$C_{21}\, C_{22}^{\;2} = 8\alpha N^{-1} \xi \; . \quad (3.2.1.35)$$

Solving Equations (3.2.1.34) and (3.2.1.35) simultaneously for C_{21} and C_{22}, we get

$$C_{21} = (8)^{\frac{1}{2}} N^{-\frac{1}{2}} \xi \left\{1 - \tfrac{2}{3}\xi\right\}^{\frac{1}{2}} \left\{\tfrac{2}{21}\xi N^{-1} + Pr\right\}^{-\frac{1}{2}} \left\{\frac{g\beta (T_w - T_\infty)}{\nu^2}\right\}^{\frac{1}{2}} \nu \quad (3.2.1.36)^*$$

and $$C_{22} = (8)^{\frac{1}{4}} N^{-\frac{1}{4}} \left\{1 - \tfrac{2}{3}\xi\right\}^{-\frac{1}{4}} \left\{\tfrac{2}{21}\xi N^{-1} + Pr\right\}^{\frac{1}{4}} \left\{\frac{g\beta (T_w - T_\infty)}{\nu^2}\right\}^{-\frac{1}{4}} (Pr)^{-\frac{1}{2}} \; . \quad (3.2.1.37)^*$$

The resultant expressions for u_x, δ_h and δ_t are

$$u_x = (8)^{\frac{1}{2}} N^{-\frac{1}{2}} \xi \nu \left\{1 - \tfrac{2}{3}\xi\right\}^{\frac{1}{2}} \left\{\tfrac{2}{21}\xi N^{-1} + Pr\right\}^{-\frac{1}{2}} \left\{\frac{g\beta (T_w - T_\infty)}{2}\right\}^{\frac{1}{2}} x^{\frac{1}{2}} \; , \quad (3.2.1.38)$$

$$\frac{\delta_h}{x} = (8)^{\frac{1}{4}} N^{-\frac{1}{4}} \left\{1 - \tfrac{2}{3}\xi\right\}^{-\frac{1}{4}} \left\{\tfrac{2}{21}\xi N^{-1} + Pr\right\}^{\frac{1}{4}} Gr_x^{\;-\frac{1}{4}}\, Pr^{-\frac{1}{2}} \; , \quad (3.2.1.39)$$

and

$$\frac{\delta_t}{x} = (8)^{\frac{1}{4}} N^{-\frac{1}{4}} \xi^{-1} \left\{1 - \tfrac{2}{3}\xi\right\}^{-\frac{1}{4}} \left\{\tfrac{2}{21}\xi N^{-1} + Pr\right\}^{\frac{1}{4}} Gr_x^{\;-\frac{1}{4}}\, Pr^{-\frac{1}{2}} \; . \quad (3.2.1.40)$$

---

*The details of the derivation of Equations (3.2.1.36) and (3.2.1.37) are given in Appendix 11.3.

The velocity and temperature profiles in the neighborhood of a plate at uniform surface heat flux are qualitatively illustrated in Fig. 3 for the cases of Pr > 1 and Pr < 1. In this analysis, the following polynomial expressions will be used to approximate the profiles:

$$u = u_x \left(\frac{y}{\delta_h}\right) \left(1 - \frac{y}{\delta_h}\right)^2 \qquad (3.2.2.1)$$

and

$$T - T_\infty = \frac{q\,\delta_t}{2k} \left(1 - \frac{y}{\delta_t}\right)^2 \qquad (3.2.2.2)$$

where the velocity u_x and the boundary-layer thicknesses, δ_h and δ_t, are functions of x and Pr, and remain to be determined.

The temperature and velocity profiles expressed by Equations (3.2.2.1) and (3.2.2.2) satisfy the following boundary conditions:

At $y = 0$; $u = 0$, and $q = -k \left(\frac{\partial T}{\partial y}\right)_{y=0}$

At $y = \delta_h$; $u = 0$, and $\frac{\partial u}{\partial y} = 0$

At $y = \delta_t$; $T = T_\infty$, and $\frac{\partial T}{\partial y} = 0$

Case 3. High Prandtl Number, Pr > 1.

The viscous and thermal effects in this case will follow a pattern similar to that discussed in Case 1; therefore, the velocity boundary-layer is considered to be thicker than the thermal boundary-layer.

Substituting Equations (3.2.2.1) and (3.2.2.2) into the integral momentum equation (3.1.1) yields

$$\frac{1}{105}\frac{d}{dx}\left(\delta_h \cdot u_x^{\ 2}\right) = \frac{1}{6}\, g\beta\left(\frac{q}{k}\right)\zeta^2\,\delta_h^{\ 2} - \nu\,\frac{u_x}{\delta_h}\,. \qquad (3.2.2.3)^*$$

Similarly, substituting Equations (3.2.2.1) and (3.2.2.2) into the integral energy equation (3.1.2) gives

$$\frac{1}{6}\frac{d}{dx}\left(M\zeta\cdot\delta_t^{\ 2}\,u_x\right) = \alpha \qquad (3.2.2.4)^*$$

where

$$M = \frac{1}{4} - \frac{1}{5}\zeta + \frac{1}{20}\zeta^2\,.$$

In order to solve the two differential equations (3.2.2.3) and (3.2.2.4), assumptions similar to those made in Case 1 are introduced here. Thus,

$$u_x = C_{31}\,x^{m_3} \qquad (3.2.2.5)$$

and $$\delta_h = C_{32}\,x^{n_3} \qquad (3.2.2.6)$$

Introducing Equations (3.2.2.5) and (3.2.2.6) into Equations (3.2.2.3) and (3.2.2.4), we get

$$\frac{2m_3+n_3}{105}\,C_{31}^{\ 2}\,C_{32}\,x^{2m_3+n_3-1} = \frac{1}{6}\,g\beta\left(\frac{q}{k}\right)C_{32}^{\ 2}\,x^{2n_3}\,\zeta^2 - C_{31}\,C_{32}^{\ -1}\,x^{m_3-n_3}\,\nu \qquad (3.2.2.7)$$

and

$$\frac{m_3+2n_3}{6}\,C_{31}\,C_{32}^{\ 2}\,M\zeta^3\,x^{m_3+2n_3-1} = \alpha\,. \qquad (3.2.2.8)$$

---

*The details of the derivation of Equations (3.2.2.3) and (3.2.2.4) are given in Appendix III.1 and III.2 respectively.

The value of the exponent of x must be the same on both sides of Equations (3.2.2.7) and (3.2.2.8) so that

$$2m_3 + n_3 - 1 = 2n_3 = m_3 - n_3$$

and $$m_3 + 2n_3 - 1 = 0$$

Solving for m_3 and n_3, we get

$$m_3 = \frac{3}{5}, \quad \text{and} \quad n_3 = \frac{1}{5}.$$

Substituting these values of m_3 and n_3 into Equations (3.2.2.7) and (3.2.2.8) yields

$$C_{31}^{\ 2}\, C_{32} = 75\left\{\frac{1}{6}\, g\beta\, (\frac{q}{k})\, C_{32}^{\ 2}\, \zeta^2 - C_{31}\, C_{32}^{\ -1}\nu\right\} \tag{3.2.2.9}$$

and

$$C_{31}\, C_{32}^{\ 2} = 6\alpha\, (M\zeta^3)^{-1} \ . \tag{3.2.2.10}$$

Solving Equations (3.2.2.9) and (3.2.2.10) simultaneously for C_{31} and C_{32}, we get

$$C_{31} = 36^{\frac{1}{10}}\, M^{-\frac{3}{5}}\, \zeta^{-1}\, \nu\left\{\frac{2}{25}\zeta^{-3}\, M^{-1} + Pr\right\}^{-\frac{2}{5}}\left\{\frac{g\beta}{\nu^2}\, (\frac{q}{k})\right\}^{\frac{2}{5}}\, Pr^{-\frac{1}{5}} \tag{3.2.2.11}^{**}$$

and

$$C_{32} = 36^{\frac{1}{5}}\, M^{-\frac{1}{5}}\, \zeta^{-1}\left\{\frac{2}{25}\zeta^{-3}\, M^{-1} + Pr\right\}^{\frac{1}{5}}\left\{\frac{g\beta}{\nu^2}\, (\frac{q}{k})\right\}^{-\frac{1}{5}}\, Pr^{-\frac{2}{5}} \ . \tag{3.2.2.12}^{**}$$

The resultant expressions for u_x, δ_h and δ_t are

**The details of derivation of Equations (3.2.2.11) and (3.2.2.12) are given in Appendix III.3.

$$u_x = 36^{\frac{1}{10}} M^{-\frac{3}{5}} \zeta^{-1} \nu \left\{ \frac{2}{25} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{2}{5}} \left\{ \frac{g\beta}{\nu^2} \left(\frac{q}{k}\right) \right\}^{\frac{2}{5}} Pr^{-\frac{1}{5}} x^{\frac{3}{5}} , \quad (3.2.2.13)$$

$$\frac{\delta_h}{x} = 36^{\frac{1}{5}} M^{-\frac{1}{5}} \zeta^{-1} \left\{ \frac{2}{25} \zeta^{-3} M^{-1} + Pr \right\}^{\frac{1}{5}} (Gr_x^*)^{-\frac{1}{5}} Pr^{-\frac{2}{5}} , \quad (3.2.2.14)$$

and

$$\frac{\delta_t}{x} = 36^{\frac{1}{5}} M^{-\frac{1}{5}} \left\{ \frac{2}{25} \zeta^{-3} M^{-1} + Pr \right\}^{\frac{1}{5}} (Gr_x^*)^{-\frac{1}{5}} Pr^{-\frac{2}{5}} , \quad (3.2.2.15)$$

where Gr_x^* is the modified Grashof Number defined as follows

$$Gr_x^* = \frac{g\beta q x^4}{\nu^2 k}$$

Surface Temperature Variation

When a value for the uniform surface heat flux is specified, the surface temperature variation can be calculated.

From Equation (3.2.2.2), the surface temperature variation with x can be expressed by

$$T_w - T_\infty = \left(\frac{q}{2k}\right) \cdot \delta_t \, . \quad (3.2.2.16)$$

Substituting Equation (3.2.2.15) into Equation (3.2.2.16), we get

$$T_w - T_\infty = \left(\frac{9}{8}\right)^{\frac{1}{5}} \left(\frac{qx}{k}\right) \left\{ \frac{0.8 \, (10 \, M)^{-1} \zeta^{-3} + Pr}{Pr^2 \, (Gr_2^*) \, M} \right\}^{\frac{1}{5}} . \quad (3.2.2.17)^{**}$$

** Details of the derivation of Equation (3.2.2.17) are given in Appendix III.4.

Equation (3.2.2.17) can be rewritten in the form

$$\frac{T_w - T_\infty}{(\frac{qx}{k})} (Gr_x^*)^{\frac{1}{5}} = (\frac{9}{8})^{\frac{1}{5}} \left\{\frac{0.8\ (10\ M)^{-1} \zeta^{-3} + Pr}{Pr^2\ M}\right\}^{\frac{1}{5}} . \qquad (3.2.2.18)$$

From Equations (3.2.2.17) and (3.2.2.18), it can be seen that the temperature difference between the wall and the ambient, $T_w - T_\infty$, is proportional to the fifth root of x. Therefore, the surface temperature variation with distance along the plate is expressed by the formula,

$$\frac{(T_w - T_\infty)_x}{(T_w - T_\infty)_L} = (\frac{x}{L})^{\frac{1}{5}} , \qquad (3.2.2.19)$$

where L is the length of the plate surface for which the flow is laminar.

By definition, the local Nusselt number is

$$Nu_x = \frac{h_x x}{k} = \left\{\frac{q}{(T_w - T_\infty)}\right\} (\frac{x}{k}) . \qquad (3.2.2.20)$$

Substituting Equation (3.2.2.18) into (3.2.2.20) leads to

$$Nu_x = (\frac{8}{9})^{\frac{1}{5}} \left\{\frac{M \cdot Pr^2}{0.8\ \zeta^{-3}\ (10\ M)^{-1} + Pr}\right\}^{\frac{1}{5}} (Gr_x^*)^{\frac{1}{5}} \qquad (3.2.2.21)$$

Rearranging of Equation (3.2.2.21) yields

$$\frac{Nu_x}{(Gr_x^*)^{\frac{1}{5}}} = (\frac{8}{9})^{\frac{1}{5}} \left\{\frac{M \cdot Pr^2}{0.8\ \zeta^{-3}\ (10\ M)^{-1} + Pr}\right\}^{\frac{1}{5}} . \qquad (3.2.2.22)$$

Equation (3.2.2.22) will be discussed later.

Case 4. Low Prandtl Number, Pr < 1.

Referring to Fig. 3(b). Let T_s be the temperature at the edge of the velocity boundary-layer.

The integral momentum equation for this case takes the form of Equation (3.2.1.26), i.e.,

$$\frac{d}{dx}\int_0^{\delta_h} u^2\, dy = -\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} + g\beta\int_0^{\delta_h}(T - T_s)\, dy \qquad (3.2.2.23)$$

The integral energy equation takes the form of Equation (3.2.1.27), i.e.,

$$\frac{d}{dx}\int_0^{\delta_h}(T - T_\infty)\, u\, dy = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \qquad (3.2.2.24)$$

Substituting Equations (3.2.2.1) and (3.2.2.2) into Equations (3.2.2.23) and (3.2.2.24) yields

$$\frac{1}{105}\frac{d}{dx}(\delta_h^{\,2}\cdot u_x^{\,2}) = (\tfrac{1}{2})g\beta\,(\tfrac{q}{k})\,\delta_h^{\,2}\,(1 - \tfrac{2}{3}\xi) - \nu\frac{u_x}{\delta_h} \qquad (3.2.2.25)^*$$

and

$$\frac{1}{6}\frac{d}{dx}\left\{N\xi^{-1}\delta_h^{\,2}u_x\right\} = \alpha\ , \qquad (3.2.2.26)^*$$

where

$$N = \frac{1}{4} - \frac{1}{5}\xi + \frac{1}{20}\xi^2$$

and

$$\xi = \frac{\delta_h}{\delta_t}\ .$$

---

*Derivations of Equations (3.2.2.25) and (3.2.2.26) are given in Appendices IV.1 and IV.2, respectively.

By a procedure similar to that employed in Case 1 of Section 3.2.2, both u_x and δ_h are assumed to follow an exponential variation with x. Thus,

$$u_x = C_{41}\, x^{m_4} \tag{3.2.2.27}$$

and

$$\delta_h = C_{42}\, x^{n_4} \ . \tag{3.2.2.28}$$

Introducing Equations (3.2.2.27) and (3.2.2.28) into Equations (3.2.2.25) and (3.2.2.26), we get

$$\frac{2m_4+n_4}{105} C_{41}^{\ 2} C_{42}\, x^{2m_4+n_4-1} = \frac{1}{2} g\beta(\frac{q}{k})\, C_{42}^{\ 2}\, x^{2n_4}\left\{1 - \frac{2}{3}\xi\right\} - C_1\, C_2^{-1}\, x^{m_4-n_4}\, \nu \tag{3.2.2.29}$$

and

$$\frac{m_4+2n_4}{6} C_{41}\, C_{42}^{\ 2}\, x^{m_4+2n_4-1}\, \xi^{-1}\, N = \alpha \ . \tag{3.2.2.30}$$

The value of the exponent of x must be the same on both sides of Equations (3.2.2.29) and (3.2.2.30); therefore,

$$2m_4 + n_4 - 1 = 2n_4 = m_4 - n_4$$

and

$$m_4 + 2n_4 - 1 = 0 \ .$$

Solving for m_4 and n_4, we get

$$m_4 = \frac{3}{5} \ , \quad \text{and} \quad n_4 = \frac{1}{5} \ .$$

Substituting these values of m_4 and n_4 back into Equations (3.2.2.29) and (3.2.2.30), we get

$$C_{41}^{\;2}\, C_{42} = 75\left\{\frac{1}{2}\, g\,\beta\,(\frac{g}{k})\, C_{42}^{\;2}\,(1 - \frac{2}{3}\xi) - C_{41}\, C_{42}^{\;-1}\,\nu\right\} \quad (3.2.2.31)$$

and

$$C_{41}\, C_{42}^{\;2} = 6\,\alpha\, N^{-1}\,\xi\;. \quad (3.2.2.32)$$

Again, solving Equations (3.2.2.31) and (3.2.2.32) simultaneously for C_{41} and C_{42},

$$C_{41} = 54^{\frac{1}{5}}\, N^{-\frac{3}{5}}\, \xi^{\frac{3}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{\frac{2}{5}} \nu \left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{-\frac{2}{5}} \left\{\frac{g\beta}{\nu^2}\,(\frac{g}{k})\right\}^{\frac{2}{5}} Pr^{-\frac{1}{5}}$$

(3.2.2.33)**

and

$$C_{42} = 12^{\frac{1}{5}}\, N^{-\frac{1}{5}}\, \xi^{\frac{1}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{-\frac{1}{5}} \left\{\frac{5}{25}\xi N^{-1} + Pr\right\}^{\frac{1}{5}} \left\{\frac{g\beta}{\nu^2}\,(\frac{g}{k})\right\}^{-\frac{1}{5}} Pr^{-\frac{2}{5}}\;.$$

(3.2.2.34)**

The resulting expressions for u_x, δ_h and δ_t are

$$u_x = 54^{\frac{1}{5}}\, N^{-\frac{3}{5}}\, \xi^{\frac{3}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{\frac{2}{5}} \nu \left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{-\frac{2}{5}} \left\{\frac{g\beta}{\nu^2}\,(\frac{g}{k})\right\}^{\frac{2}{5}} Pr^{-\frac{1}{5}}\, x^{\frac{3}{5}}\;,$$

(3.2.2.35)

$$\frac{\delta_h}{x} = 12^{\frac{1}{5}}\, N^{-\frac{1}{5}}\, \xi^{\frac{1}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{-\frac{1}{5}} \left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{\frac{1}{5}} (Gr_x^{\;*})^{-\frac{1}{5}}\, Pr^{-\frac{2}{5}}\;, \quad (3.2.2.36)$$

and

**Details of the derivation of Equations (3.2.2.33) and (3.2.2.34) are given in Appendix III.3.

$$\frac{\delta_t}{x} = 12^{\frac{1}{5}} N^{-\frac{1}{5}} \xi^{-\frac{4}{5}} \left\{1 - \frac{2}{3}\xi\right\}^{-\frac{1}{5}} \left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{\frac{1}{5}} (Gr_x{}^*)^{-\frac{1}{5}} Pr^{\frac{2}{5}} . \quad (3.2.2.37)$$

Surface Temperature Variation

By a procedure similar to that employed in Case 3, the surface temperature variation can be calculated from Equation (3.2.2.2) by letting $y \to 0$; thus,

$$T_w - T_\infty = \frac{q}{2k} \delta_t . \quad (3.2.2.38)$$

Substituting Equation (3.2.2.37) into Equation (3.2.2.28) yields

$$T_w - T_\infty = (\tfrac{3}{8})^{\frac{1}{5}} (\tfrac{qx}{k}) \left\{\frac{0.8\ (10\ N)^{-1}\ \xi + Pr}{Pr^2\ Gr_x{}^* N\ \xi^4\ (1 - \frac{2}{3}\xi)}\right\}^{\frac{1}{5}} , \quad (3.2.2.39)^{**}$$

where $Gr_x{}^*$ is the modified Grashof number.

Equation (3.2.2.39) can be rewritten in the form,

$$\frac{T_w - T_\infty}{\frac{qx}{k}} (Gr_x{}^*)^{\frac{1}{5}} = (\tfrac{3}{8})^{\frac{1}{5}} \left\{\frac{0.8\ (10\ N)^{-1}\ \xi + Pr}{Pr^2\ N\ \xi^4\ (1 - \frac{2}{3}\xi)}\right\}^{\frac{1}{5}} . \quad (3.2.2.40)$$

From Equations (3.2.2.39) and (3.2.2.40), it can be seen that the temperature difference between the wall surface and the ambient, $T_w - T_\infty$, is proportional to the fifth root of x.

The surface temperature variation with distance along the plate is

---

** Details of the derivation of Equation (3.2.2.39) are given in Appendix III.4.

$$\frac{(T_w - T\ )_x}{(T_w - T\ )_L} = (\frac{x}{L})^{\frac{1}{5}} \tag{3.2.2.41}$$

By a procedure similar to that employed in Case 3, the local Nusselt number can be written as

$$Nu_x = (\frac{8}{3})^{\frac{1}{5}} \left\{ \frac{N \xi^4\ (1 - \frac{2}{3}\xi)\ Pr^2}{0.8\,\xi\,(10\ N)^{-1} + Pr} \right\}^{\frac{1}{5}} (Gr_x{}^*)^{\frac{1}{5}}\ . \tag{3.2.2.42}$$

Rearranging Equation (3.2.2.42) yields

$$\frac{Nu_x}{(Gr_x{}^*)^{\frac{1}{5}}} = (\frac{8}{3})^{\frac{1}{5}} \left\{ \frac{N \xi^4\ (1 - \frac{2}{3}\xi)\ Pr^2}{0.8\,\xi\,(10\ N)^{-1} + Pr} \right\}^{\frac{1}{5}}\ . \tag{3.2.2.43}$$

Equation (3.2.2.43) will be discussed later.

3.3. LOCAL NUSSELT NUMBER CALCULATIONS

In order to be able to calculate the local Nusselt number for the four cases discussed in the previous section, it is necessary to evaluate the parameters ζ and ξ in Equations (3.2.1.25), (3.2.1.43), (3.2.2.22), and (3.2.2.43). In the analysis, it was assumed that ζ and ξ were only functions of Pr. For simplicity it was assumed that

$$\zeta = \frac{1}{(Pr)^{\frac{1}{\gamma}}} \qquad \text{for} \quad Pr > 1$$

$$\xi = (Pr)^{\frac{1}{\gamma}} \qquad \text{for} \quad Pr < 1$$

where γ is a parameter to be determined.

In Equation (3.2.1.25), τ was selected so that the prediction of $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ by that equation will match the prediction of the exact solution of Ostrach (9) at corresponding values of Pr. It is to be recalled that Equation (3.2.1.25) is applicable to fluids of $Pr > 1$, when the wall temperature is constant. The values of Pr selected for the matching process were 1, 10, 100 and 1000. Over this range of Pr, it was found that τ could be approximated empirically by

$$\tau = (Pr)^{0.42Pr^{-0.174}} \quad . \tag{3.3.1}$$

Equation (3.3.1) is plotted in Fig. 4. The prediction of $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ given by Equations (3.2.1.25) and (3.3.1) is shown in Fig. 5. The results of Ostrach (9) and the approximate solution of Eckert (2) are also plotted on the same figure for comparison.

Fig. 4 shows that τ varies from 1, at $Pr = 1$, to 2.39 at $Pr = 1000$. Also, the change in the value of τ is negligible in the range of Pr between 100 and 1000. As anticipated, Fig. 5 shows that the prediction of $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ of the present analysis is in fair agreement with the prediction of Ostrach (9) except for values of Pr between 1 and 4. In general, the approximate analysis of Eckert (2) predicts higher values of $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ over the whole range of Pr. This might be due to the fact that, in Eckert's analysis, it was assumed that $\delta_h = \delta_t$.

In a similar manner, the value of the parameter γ in Equation (3.2.2.22) was determined by matching its prediction for $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ with

the prediction of the exact solution of Sparrow and Gregg (15). Values of Pr selected for the matching process were 1, 10, 100. It was found that the same empirical relation for τ given by Equation (3.3.1) and plotted in Fig. 4 is also applicable to this case. This means that the ratio of the thicknesses of the thermal boundary-layer to the velocity boundary-layer does not depend on the boundary condition at the plate surface for fluids of $Pr > 1$. It is to be recalled that this is the case of fluids of $Pr > 1$ with constant wall flux at the plate wall.

Figure 6 shows the prediction of $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ by Equation (3.2.2.22) for various Prandtl numbers. On the same Figure, results of Sparrow and Gregg (15) as well as the results of the approximate solution of Sparrow (14) are also shown. Despite the fact that the selection of τ in this analysis was based on matching the prediction of Equation (3.2.2.22) with the prediction of the exact solution of Sparrow and Gregg, at Prandtl numbers 1, 10 and 100, the agreement between the prediction of the present analysis and that of Sparrow and Gregg for $10 < Pr < 1000$ is fairly good. Figure 6 also shows that the approximate analysis of Sparrow (14) predicts higher values for $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ over the whole range of Pr. This also may be due to the fact that Sparrow assumed in his analysis that $\delta_h = \delta_t$.

For fluids of $Pr < 1$, the exact solution of Ostrach is available only for Prandtl numbers of 0.733 and 0.01. Following the same procedure as was employed with Equations (3.2.1.25) and (3.2.2.22) in determining τ, it was found that the value of τ in Equation (3.2.1.43) must assume the form

$$\tau = (Pr)^{-0.855Pr^{0.027}} \qquad (3.3.2)$$

Figure 7 shows a plot of Equation (3.3.2). Equation (3.3.2) was used in evaluating $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ in Equation (3.2.1.43). The results are plotted in Fig. 8. On the same Figure, the results of the exact solution of Ostrach (9) as well as the results of the approximate solution of Eckert (2) are also shown. Values of $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ from the exact solution of Ostrach in the range of $0.01 < Pr < 1$ are plotted in Fig. 8. The results of the present analysis are slightly lower than the prediction of Ostrach for $Pr < 0.1$. The approximate analysis of Eckert (2) predicts higher values of $\frac{Nu_x}{(Gr_x/4)^{\frac{1}{4}}}$ for $Pr > 0.06$ and lower values for $Pr < 0.06$.

For the last case, namely, the case of natural convection about a vertical plate with constant heat flux for fluids of $Pr < 1$, Sparrow's paper (15) reported only the results of the exact solution for only one Prandtl number, namely, 0.1. Due to lack of information needed for this case, it was assumed that γ as used in Equation (3.2.2.43) could be determined by Equation (3.3.2). If this is the case, one can assume that, for fluids of $Pr < 1$, the ratio of the thickness of the velocity boundary-layer to the thickness of the thermal boundary-layer is independent of the boundary condition at the wall. Results of the prediction of $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ given by Equation (3.2.2.43) are shown in Fig. 9. On the same Figure, results of the approximate solution of Sparrow are also shown for the sake of comparison.

The curve of heat transfer coefficient predicted by the exact solution of Sparrow and Gregg (15) was drawn by extrapolation of their results from $Pr > 1$. Figure 9 shows that the approximate analysis of Sparrow (14)

predicts values of $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ slightly higher than those of the present analysis for values of $Pr > 0.03$ and values lower than those of the present analysis for $Pr < 0.03$.

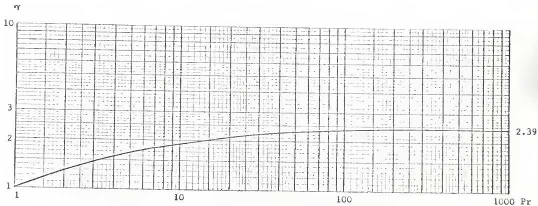


Fig. 4. Values of parameter γ for $Pr > 1$

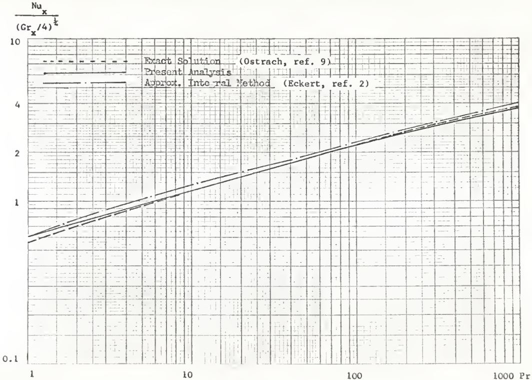


Fig. 5. Dimensionless heat transfer coefficient as a function of Pr number for the case of uniform surface temperature and Pr > 1.

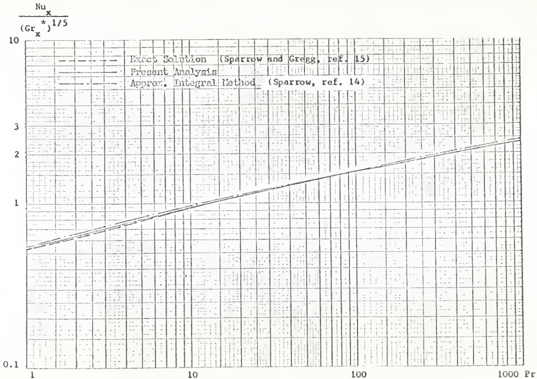


Fig. 6. Dimensionless heat transfer coefficient as a function of Pr number for the case of uniform surface heat flux and Pr > 1.

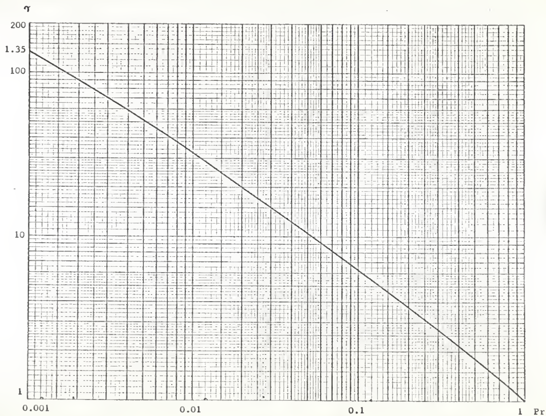


Fig. 7. Values of parameter γ for $Pr < 1$.

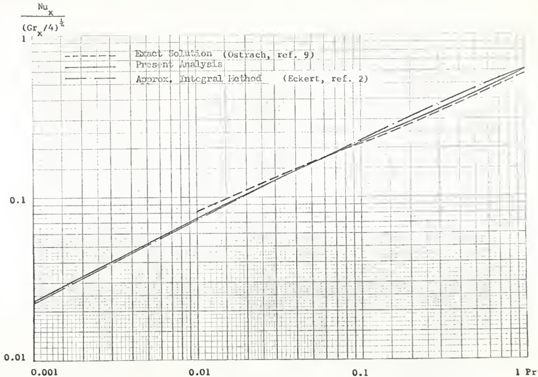


Fig. 8. Dimensionless heat transfer coefficient as a function of Pr number for the case of uniform surface temperature and Pr < 1.

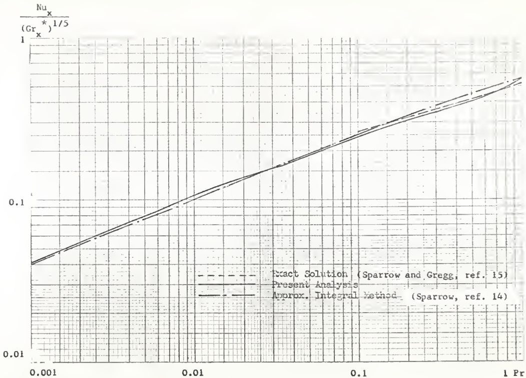


Fig. 9. Dimensionless heat transfer coefficient as a function of Pr number for the case of uniform surface heat flux and Pr < 1.

CHAPTER 4

CONCLUSIONS

The problem of laminar free-convection on a vertical plate with uniform wall temperature or uniform heat flux at the wall was investigated by applying the Karman-Pohlhausen method. The analysis was made for fluids whose Prandtl numbers are either greater than, less than, or equal to one. Unlike previous investigations, the assumption that the thicknesses of the velocity and the thermal boundary layers are identical was not made. In the analysis, it was assumed that the ratio of the thicknesses is a function of Prandtl number only, i.e., it was assumed that

$$\frac{\delta_t}{\delta_h} = \zeta = \frac{1}{Pr^{\frac{1}{\gamma}}} \qquad \text{for fluids of } Pr > 1$$

and that

$$\frac{\delta_h}{\delta_t} = \xi = Pr^{\frac{1}{\gamma}} \qquad \text{for fluids of } Pr < 1 \ .$$

The resulting expressions for Nusselt number, for the four cases analyzed required a knowledge of the parameter γ . γ was determined by matching the theoretical predictions in this analysis with the prediction of known exact solutions at corresponding Prandtl numbers. From the results of the investigation, the following conclusions can be drawn:

1. For the case of a vertical plate with uniform wall temperature for fluids of $Pr > 1$, the parameter γ is a function of Pr. The prediction

of $\frac{Nu_x}{(Gr_x/4)^{1/4}}$ of this analysis is slightly lower than the prediction of Eckert's approximate analysis over the range of Prandtl number investigated, namely, $1 < Pr < 1000$.

2. For the case of a vertical plate with uniform heat flux, for fluids of $Pr > 1$, the parameter Υ was identical to that of the previous case. This means that the ratio ξ is independent of the boundary condition at the wall. The prediction of $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ of the present analysis is lower than the prediction made on the basis of Sparrow's analysis (14).

3. For fluids of $Pr < 1$, in the case of a vertical plate with uniform wall temperature, the parameter Υ is also a function of Pr. The approximate analysis of Eckert (2) predicts higher values of $\frac{Nu_x}{(Gr_x/4)^{1/4}}$ for fluids of $Pr > 0.06$ and lower values for $Pr < 0.06$, than does the present analysis.

4. For fluids of $Pr < 1$, in the case of vertical plate with uniform heat flux, it was assumed that Υ is identical to its value in the case of the plate at constant wall temperature. If this is the case, the ratio of the thickness of the velocity boundary-layer to the thickness of the thermal boundary-layer is independent of the boundary condition at the wall. For this case, the approximate analysis of Sparrow (14) predicts values of $\frac{Nu_x}{(Gr_x^*)^{1/5}}$ slightly higher than those of the present analysis for fluids of $Pr > 0.03$ and values less than those of the present analysis for $Pr < 0.03$.

NOMENCLATURE

Symbols

C_{i1} Constant in the equation $u_x = C_{i1} x^{m_i}$ and defined by Equations (3.2.1.14), (3.2.1.36), (3.2.2.11) and (3.2.2.33)

C_{i2} Constant in the equation $\delta_h = C_{i2} x^{n_i}$ and defined by Equations (3.2.1.15), (3.2.1.37), (3.2.2.12) and (3.2.2.34)

c_p Specific heat at constant pressure, Btu/slug deg F

g Gravitational acceleration, ft/sec^2

h_x Local film heat transfer coefficient, Btu/hr sq ft deg F

K_ε Constant in the equation $Nu_x = K_\varepsilon \, (Gr_x \, Pr^\varepsilon)^{\frac{1}{4}}$

k Thermal conductivity, Btu/hr ft sq deg F

L Length of the flat plate over which the flow is laminar, ft

M Function defined by the equation $M = \frac{1}{4} - \frac{1}{5}\xi + \frac{1}{20}\xi^2$, dimensionless

m_i Exponent in the equation $u_x = C_{i1} x^{m_i}$, dimensionless

m' Parameter defined by the equation $m' = s^2$, dimensionless

N Function defined by the equation $N = \frac{1}{4} - \frac{1}{5}\xi + \frac{1}{20}\xi^2$, dimensionless

n_i Exponent in the equation $\delta_h = C_{i2} x^{n_i}$, dimensionless

q Heat flux rate at the wall, Btu/hr sq ft

s Parameter in the approximate velocity profile suggested by Fujii, dimensionless

T Static temperature, deg F

T_s Temperature at the edge of the velocity boundary-layer, deg F

T_w Wall temperature, deg F

T_∞ Ambient temperature, deg F

u	Velocity component in x-direction, ft/sec
u_x	Characteristic velocity, ft/sec
v	Velocity component in y-direction, ft/sec
x	Distance measured along the flat plate from the leading edge, ft
y	Distance measured normal to the flat plate, ft
α	Thermal diffusivity, ft^2/sec
β	Coefficient of volumetric expansion, $-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p$, deg R^{-1}
γ	Parameter in the equation $\zeta = \frac{1}{Pr^{1/\gamma}}$, dimensionless
δ	Boundary-layer thickness when $\delta_h = \delta_t = \delta$, ft
δ_h	Velocity boundary-layer thickness, ft
δ_t	Thermal boundary-layer thickness, ft
ε	Parameter in equation $Nu_x = K_\epsilon \, (Gr_x \cdot Pr^\epsilon)^{\frac{1}{4}}$ suggested by Yamagata
ζ	Ratio of thermal boundary-layer thickness to velocity boundary-layer thickness, $\zeta = \frac{\delta_t}{\delta_h}$, dimensionless
η	Similarity variable, $\eta = \frac{y}{\delta}$, dimensionless
θ	Dimensionless temperature profile, $\theta = \frac{T - T_\infty}{T_w - T_\infty}$
μ	Absolute viscosity, slug/ft sec
ν	Kinematic viscosity, ft^2/sec
ξ	Ratio of velocity boundary-layer thickness to thermal boundary-layer thickness, $\xi = \frac{\delta_h}{\delta_t}$, dimensionless
ρ	Density, slug/ft^3
τ_{yx}	x-direction tangential shear stress on the flat plate surface, lbf/ft^2
ϕ	Dimensionless velocity profile, $\phi = \frac{u}{u_x}$

Dimensionless Groups

Gr_x Grashof number based on x, $\frac{g\beta\,(T - T_\infty)\,x^3}{\nu^2}$

Gr_x^* Modified Grashof number based on x, $\frac{g\beta\,qx^4}{k\nu^2}$

Nu_x Local Nusselt number based on x, $\frac{h_x x}{k}$

Pr Prandtl number, $\frac{C_p\mu}{k} = \frac{\nu}{\alpha}$

Subscripts

h Denotes velocity field

i Denotes case number of the problems

t Denotes thermal field

w Denotes the wall

∞ Denotes evaluation at undisturbed conditions

ACKNOWLEDGMENTS

The author wishes to express his sincere gratitude to his major advisor, Dr. N. Z. Azer, for his constant encouragement, advice, and valuable suggestions. The author also expresses his gratitude to Dr. R. G. Nevins, Head of the Department of Mechanical Engineering, Dr. J. M. Bowyer, Jr., Professor of Mechanical Engineering, and Dr. C. J. Hsu, Professor of Mathematics, for being members of the author's advisory committee.

BIBLIOGRAPHY

1. Brindley, J. "An Approximation Technique for Natural Convection in a Boundary-Layer." Int. J. Heat Mass Transfer, vol. 6 (1963), 1035-1948.

2. Eckert, E. R. G., and Drake, R. M. "Heat and Mass Transfer." McGraw-Hill, 2nd ed. (1959), 311-333.

3. Eckert, E. R. G., and Soehengen, E. E. "Studies on Heat Transfer in Laminar Free Convection with the Zehnder-Mach Interferometer." Tech. Rep. No. 5747, A.T.I. No. 44580, Air Material Command, Dayton, Ohio (Dec. 27, 1948).

4. Fujii, T. "Mathematical Analysis of Heat Transfer from a Vertical Flat Surface by Laminar Free Convection." Bulletin of Japanese Society of Mechanical Engineering, vol. 2, No. 7 (1959).

5. Howarth, L., ed. "Modern Developments in Fluid Dynamics, High Speed Flow." vol. II. Clarendum Press, Oxford, England (1953).

6. Lorenz, H. H. "Die Warmeübertragung an einer ebenen senkrechten Platte an Öl bei natürlicher Konvektion." Z. Techn, Physik (1934), 362.

7. Meksyn, D. "New Methods in Laminar Boundary Layer Theory." Pergamon, London (1961).

8. Merk, H. J., and Prins, J. A. "Thermal Convection Laminar Boundary-Layer, I." Applied Sci. Res. (A), 4 (1953-1954), 11-24.

9. Ostrach, S. "An Analysis of Laminar Free Convection Flow and Heat Transfer about a Flat Plate Parallel to the Direction of the Generating Body Force." NACA TR 1111 (1953).

10. Pohlhausen, E. "Der Wärmeanbtausch Zwichen festen Körpern und Flüssigkeiten mit kleiner Reiburg und kleiner Wärmeleitung." ZAAM, 1 (1921), 115.

11. Saunders, O. A. "Natural Convection in Liquid." Proceedings of the Royal Society of London, England, Series A, vol. 172 (1939), 55-71.

12. Schmidt, E., and Beckmann, W. "Das Temperature und Geschwindigkeitsfeld vor einer Wärme abgebenden senkrechter Platte bei naturalicher Konvektion." Tech. Mech. Thermodynamik, 1 (1930), 341-349, 391-406.

13. Schuh, H. "Boundary-Layers of Temperature." Reps. & Trans. 1007, AVA Monographs, British M.A.P. (April 15, 1948).

14. Sparrow, E. M. "Laminar Free Convection on a Vertical Plate with Prescribed Nonuniform Wall Heat Flux or Prescribed Nonuniform Wall Temperature." NACA TN 3508 (1955).

15. Sparrow, E. M., and Gregg, J. L. "Laminar Free Convection from a Vertical Plate with Uniform Surface Heat Flux." ASME Trans., 78, (Feb. 1956), 435-440.

16. Yamagata, K. "An Analysis of Free-Convection about a Vertical Plate," Trans. Japan Society of Mech. Eng., Vol. 24, No. 144 (Aug. 1958), 541-546.

Other References Related to the Topic of This Report.

17. Chang, K. S., Akins, R. G., and Bankoff, S. G. "Free Convection of a Liquid Metal from a Uniformly Heated Vertical Plate." Industrial and Engineering Chemistry Fundamentals, vol. 5 (1966), 26-37.

18. Elskin, K. V. "An Approximate Solution of Equation of Free Convection of a Liquid near a Vertical Non-Isothermal Wall." Inzh-Fiz Zh., vol. 4, No. 4 (1961), 62.

19. Fritsch, C. A., and Grosh, R. J. "Free Convection Heat Transfer to a Supercritical Fluid." Argone National Laboratory ANL-6486 (Nov. 1961).

20. Kreith, F. "Principles of Heat Transfer." International Textbook (April, 1965), 326-363.

21. Kutateladze, S. S., Borishanskii, V. M., Novikov, I. I., and Fedynskii, O. S. "Liquid-Metal Heat Transfer Media." Consultants Bureau, Inc., New York (1959).

22. McAdams, W. H. "Heat Transmission." McGraw-Hill, 3rd ed. (1954), 165-183.

23. Scherberg, M. G. "Natural Convection from Wall Section of Arbitrary Distribution by an Integral Method." Int. J. of Heat Mass Transfer, Vol. 7 (1964), 501-506.

24. Schlichting, H. "Boundary-Layer Theory." McGraw-Hill, 4th ed. (1962), 332-336.

25. Touloukian, Y. S., Hawkins, G. A., and Jacob, M. "Heat Transfer by Free Convection from Heated Vertical Surfaces to Liquid." ASME Trans., 70 (Jan. 1948), 13-18.

26. Yang, K. T. "Possible Similarity Solutions from Laminar Free Convection on Vertical Plates and Cylinders." ASME Trans, 82, E (J. Appl. Mech.) (June 1960), 230-236.

APPENDICES

APPENDIX I

I.1. Derivation of Equation (3.2.1.6)

The derivation will be conducted by substituting Equations (3.2.1.1) and (3.2.1.2) into each term of Equation (3.1.1);

$$\int_0^{\delta_h} u^2 \, dy = \int_0^{\delta_h} u_x^2 \left(\frac{y}{\delta_h}\right)^2 \left(1 - \frac{y}{\delta_h}\right)^4 dy \ .$$

Let

$$\frac{y}{\delta_h} = \eta \ .$$

Then

$$dy = \delta_h \, d\eta \ .$$

This leads to

$$\int_0^{\delta_h} u^2 \, dy = \int_0^1 u_x^2 \, \eta^2 (1 - \eta)^4 \, \delta_h \, d\eta$$

$$= u_x^2 \, \delta_h \int_0^1 \eta^2 (1 - \eta)^4 \, d\eta$$

$$= \frac{1}{105} \, u_x^2 \cdot \delta_h \ .$$

Therefore,

$$\frac{d}{dx} \int_0^{\delta_h} u^2 \, dy = \frac{1}{105} \frac{d}{dx} \left(\delta_h \cdot u_x^2 \right) \ . \qquad (I.1.1)$$

The shear stress at the wall is given by

$$\tau_{yz} = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \ .$$

However,

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{ u_x \left(\frac{y}{\delta_h}\right)\left(1 - \frac{y}{\delta_h}\right)^2 \right\}$$

or

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\,\frac{d\eta}{dy} = \frac{1}{\delta_h}\cdot\frac{\partial u}{\partial \eta}$$

so that

$$\frac{\partial u}{\partial \eta} = \frac{\partial}{\partial \eta}\left\{ u_x \eta \,(1 - \eta)^2 \right\} = u_x \,(1 - 4\eta - 3\eta^2) .$$

Hence,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{1}{\delta_h}\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} ,$$

but

$$\left(\frac{\partial u}{\partial \eta}\right)_{\eta=0} = u_x ;$$

therefore,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{u_x}{\delta_h} . \qquad (I.1.2)$$

Also,

$$\int_0^{\delta_h} (T - T_\infty)\, dy = \int_0^{\delta_t} (T - T_\infty)\, dy + \int_{\delta_t}^{\delta_h} (T - T_\infty)\, dy .$$

But, for $y > \delta_t$, $T = T_\infty$; therefore,

$$\int_{\delta_t}^{\delta_h} (T - T_\infty)\, dy = 0$$

or $$\int_0^{\delta_t} (T - T_\infty)\, dy = \int_0^{\delta_t} (T_w - T_\infty)\,(1 - \frac{y}{\delta_t})^2\, dy$$

or $$(T_w - T_\infty) \int_0^{\delta_t} \left\{ 1 - \frac{2y}{\delta_t} + (\frac{y}{\delta_t})^2 \right\} dy = \frac{1}{3}\,(T_w - T_\infty)\,\delta_t\,.$$

Thus,

$$\int_0^{\delta_h} (T - T_\infty)\, dy = \frac{1}{3}\,(T_w - T_\infty)\,\delta_t\,. \qquad (I.1.3)$$

Substituting Equations (I.1.1), (I.1.2) and (I.1.3) into Equation (3.1.1), we obtain

$$\frac{d}{dx}\,(\frac{1}{105}\,u_x^2 \cdot \delta_h) = -\,\nu\,\frac{u_x}{\delta_h} + \frac{1}{3}\,g\,\beta\,(T_w - T_\infty)\,\delta_t$$

or

$$\frac{1}{105}\,\frac{d}{dx}\,(u_x^2 \cdot \delta_h) = \frac{1}{3}\,g\,\beta\,(T_w - T_\infty)\,\xi\,\delta_h - \nu\,\frac{u_x}{\delta_h}\,. \qquad (3.2.1.6)$$

I.2. Derivation of Equation (3.2.1.7)

The derivation will be conducted by substituting Equation (3.2.1.1) and (3.2.1.2) into each term of Equation (3.1.2):

$$\int_0^{\delta_t} (T - T_\infty)\, u\, dy = \int_0^{\delta_t} (T_w - T_\infty)\,(1 - \frac{y}{\delta_t})^2\, u_x\,(\frac{y}{\delta_h})\,(1 - \frac{y}{\delta_h})^2\, dy$$

$$= u_x\,(T_w - T_\infty) \int_0^{\delta_t} (\xi\,\frac{y}{\delta_t})\,(1 - \xi\,\frac{y}{\delta_t})^2\,(1 - \frac{y}{\delta_t})^2\, dy$$

$$= u_x \,(T_w - T_\infty) \int_0^{\delta_t} (\zeta \frac{y}{\delta_t}) \left\{ 1 - 2\,(1 + \zeta)\,(\frac{y}{\delta_t}) \right.$$

$$\left. + \,(1 + 4\,\zeta + \zeta^2)\,(\frac{y}{\delta_t})^2 - 2\zeta\,(1 + \zeta)\,(\frac{y}{\delta_t})^3 + \zeta^3\,(\frac{y}{\delta_t})^4 \right\} dy$$

where $\zeta = \dfrac{\delta_t}{\delta_h}$.

After integration we get

$$\int_0^{\delta_t} (T - T_\infty)\, u\, dy = u_x\,(T_w - T_\infty)\,\delta_t\,\zeta \left\{ \frac{1}{12} - \frac{1}{15}\zeta + \frac{1}{60}\zeta^2 \right\} . \qquad (I.2.1)$$

Now,

$$\frac{\partial T}{\partial y} = \frac{\partial}{\partial y} \left\{ (T_w - T_\infty)\,(1 - \frac{y}{\delta_t})^2 \right\}$$

$$= 2\,(T_w - T_\infty)\,(1 - \frac{y}{\delta_t})\,(- \frac{1}{\delta_t}) .$$

From this,

$$(\frac{\partial T}{\partial y})_{y=0} = - \frac{2}{\delta_t}\,(T_w - T_\infty) . \qquad (I.2.2)$$

Substituting Equations (I.2.1) and (I.2.2) into Equation (3.1.2) yields

$$\frac{d}{dx} \left\{ u_x \cdot \delta_t \,(T_w - T_\infty)\,(\frac{1}{12} - \frac{1}{15}\zeta + \frac{1}{60}\zeta^2)\,\zeta \right\} = \frac{2\alpha}{\delta_t}\,(T_w - T_\infty) . \qquad (I.2.3)$$

Let $\quad \frac{1}{4} - \frac{1}{5}\zeta + \frac{1}{20}\zeta^2 = M$.

As a result, Equation (I.2.3) can be rewritten as

$$\frac{1}{6} \frac{d}{dx} \left\{ M u_x \delta_t \zeta \right\} = \frac{\alpha}{\delta_t} . \tag{3.2.1.7}$$

I.3. <u>Derivation of Equations (3.2.1.14) and (3.2.1.15)</u>

From Equation (3.2.1.13), we have

$$C_{11} = 8 \alpha (C_{12}{}^2 M \zeta^3)^{-1} . \tag{I.3.1}$$

Substituting Equation (1.3.1) into Equation (3.2.1.12), we get

$$64 \alpha^2 C_{12}{}^{-3} \left\{ M \zeta^3 \right\}^{-2} = 84 \left[\frac{C_{12}}{3} g \beta (T_w - T_\infty) \zeta - 8 \alpha \nu C_{12}{}^{-3} \left\{ M \zeta^3 \right\}^{-1} \right]$$

or $$16 \alpha^2 \left\{ M \zeta^3 \right\}^{-2} = 21 \left[\frac{C_{12}{}^4}{3} g \beta (T_w - T_\infty) \zeta - 8 \alpha \nu \left\{ M \zeta^3 \right\}^{-1} \right]$$

or $$\frac{C_{12}{}^4}{3} g \beta (T_w - T_\infty) \zeta = \frac{16}{21} \alpha^2 \left\{ M \zeta^3 \right\}^{-2} + 8 \alpha \nu \left\{ M \zeta^3 \right\}^{-1}$$

Then,

$$C_{12}{}^4 = 3 \left\{ g \beta (T_w - T_\infty) \zeta \right\}^{-1} \left[8 (M \zeta^3)^{-1} \left\{ \frac{2}{21} \alpha^2 (M \zeta^3)^{-1} + \alpha \nu \right\} \right]$$

or $$C_{12} = (24)^{\frac{1}{4}} \left\{ \frac{g \beta (T_w - T_\infty)}{\nu^2} \right\}^{-\frac{1}{4}} (M \zeta^4)^{-\frac{1}{4}} \left\{ \frac{2}{21} (M \zeta^3)^{-1} + \frac{\nu}{\alpha} \right\}^{\frac{1}{4}} \left(\frac{\nu}{\alpha}\right)^{-\frac{1}{2}}$$

or $$C_{12} = (24)^{\frac{1}{4}} M^{-\frac{1}{4}} \zeta^{-1} \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{\frac{1}{4}} \left\{ \frac{g \beta (T_w - T_\infty)}{\nu^2} \right\}^{-\frac{1}{4}} (Pr)^{-\frac{1}{2}}$$

$$(3.2.1.15)$$

Substituting this value of C_{12} into Equation (I.3.1), we get

$$C_{11} = 8 \alpha (M \zeta^3)^{-1} (24)^{-\frac{1}{2}} M^{\frac{1}{2}} \zeta^2 \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{1}{2}} \left\{ \frac{g \beta (T_w - T)}{\nu^2} \right\}^{\frac{1}{2}} (Pr)$$

or

$$C_{11} = \left(\frac{8}{3}\right)^{\frac{1}{2}} M^{-\frac{1}{2}} \zeta^{-1} \left\{ \frac{2}{21} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{1}{2}} \left\{ \frac{g \beta (T_w - T_\infty)}{2} \right\}^{\frac{1}{2}} \nu \quad .$$

(3.2.1.14)

APPENDIX II

II.1. Derivation of Equation (3.2.1.28)

$$T - T_s = (T - T_\infty) - (T_s - T_\infty)$$

Substituting Equation (3.2.1.2) into the above Equation, we get

$$T - T_s = (T_w - T_\infty)\,(1 - \frac{y}{\delta_t})^2 - (T_w - T_\infty)\,(1 - \frac{\delta_h}{\delta_t})^2$$

$$= (T_w - T_\infty)\left\{2\,\xi\,(1 - \frac{y}{\delta_h}) - \xi^2\,(1 - \frac{y^2}{\delta_h^2})\right\},$$

where $\xi = \dfrac{\delta_h}{\delta_t}$.

Hence,

$$\int_0^{\delta_h}\left\{(T - T_\infty) - (T_s - T_\infty)\right\}dy = \int_0^{\delta_h}(T_w - T_\infty)\left\{2\,\xi\,(1 - \frac{y}{\delta_h})\right.$$

$$\left. - \xi^2\,(1 - \frac{y^2}{\delta_h^2})\right\}dy = (T_w - T_\infty)\;\delta_h\cdot\xi\,(1 - \frac{2}{3}\xi)\;. \qquad \text{(II.1.1)}$$

Substituting Equation (I.1.1) and (I.1.2) and (II.1.1) into Equation (3.2.1.26) yields

$$\frac{1}{105}\;\frac{d}{dx}\,(\delta_h\cdot u_x^2) = g\,\beta\,(T_w - T_\infty)\;\xi\;(1 - \frac{2}{3}\xi)\;\delta_h - \nu\,\frac{u_x}{\delta_h}\;. \qquad \text{(3.2.1.28)}$$

II.2. Derivation of Equation (3.2.1.29)

By a procedure similar to those in Appendix I.2, each term of Equation (3.2.1.27) will be evaluated:

$$\int_0^{\delta_h} (T - T_\infty)\, u\, dy = \int_0^{\delta_h} (T_w - T_\infty)(1 - \frac{y}{\delta_h})^2\, u_x\, (\frac{y}{\delta_h})(1 - \frac{y}{\delta_h})^2\, dy$$

$$= u_x\, (T_w - T_\infty) \int_0^{\delta_h} (1 - \frac{y}{\delta_h}\, \xi)^2\, (\frac{y}{\delta_h})\, (1 - \frac{y}{\delta_h})^2\, dy$$

$$= u_x\, (T_w - T_\infty) \int_0^{\delta_h} \left\{ \frac{y}{\delta_h} - 2\,(1 + \xi)(\frac{y}{\delta_h})^2 + (1 + 4\xi + \xi^2)(\frac{y}{\delta_h})^3 \right.$$

$$\left. - 2\,\xi\,(1 + \xi)\,(\frac{y}{\delta_h})^4 + \xi^2\,(\frac{y}{\delta_h})^5 \right\} dy$$

$$= u_x\, (T_w - T_\infty)\, \delta_h \left\{ \frac{1}{12} - \frac{1}{15}\,\xi + \frac{1}{60}\,\xi^2 \right\} .$$

Let

$$N = \frac{1}{4} - \frac{1}{5}\,\xi + \frac{1}{20}\,\xi^2 .$$

Thus,

$$\int_0^{\delta_h} (T - T_\infty)\, u\, dy = \frac{1}{3}\, N \cdot \delta_h\, u_x\, (T_w - T_\infty) . \qquad \text{(II.2.1)}$$

Substituting Equations (I.2.2) and (II.2.1) into Equation (3.2.1.27), we get

$$\frac{d}{dx} \left\{ \frac{1}{3}\, N \cdot \delta_h \cdot u_x \cdot (T_w - T_\infty) \right\} = \frac{2\alpha}{\delta_t}\, (T_w - T_\infty) .$$

Therefore,

$$\frac{1}{6}\ \frac{d}{dx}\left\{ N\cdot\delta_h\, u_x \right\} = \frac{\alpha}{\delta_t}\ . \tag{3.2.1.29}$$

II.3. Derivation of Equations (3.2.1.36) and (3.2.1.37)

From Equation (3.2.1.35), we get

$$C_{21} = 8\,\alpha\, C_{22}^{-2}\, N^{-1}\,\xi \tag{II.3.1}$$

or

$$C_{21}^{\ 2}\, C_{22} = (8\,\alpha\, C_{22}^{-2}\, N^{-1}\,\xi)^2\, C_{22} = 64\,\alpha^2\, C_{22}^{-3}\, N^{-2}\,\xi^2\ . \tag{II.3.2}$$

From Equations (II.3.2) and (3.2.1.34) we can write

$$64\,\alpha^2\, C_{22}^{-3}\, N^{-2}\,\xi^2 = 84\left\{ C_{22}\, g\,\beta\,(T_w - T_\infty)(\xi - \tfrac{2}{3}\,\xi^2) - 8\,\alpha\,\nu\, C_{22}^{-3}\, N^{-1}\,\xi \right\}$$

or

$$\frac{16}{21}\,\alpha^2\, N^{-2}\,\xi = C_{22}^{\ 4}\, g\,\beta\,(T_w - T_\infty)\ (1 - \tfrac{2}{3}\,\xi) - 8\,\alpha\,\nu\, N^{-1}\ .$$

Thus,

$$C_{22}^{\ 4} = \left\{ g\,\beta\,(T_w - T_\infty)\ (1 - \tfrac{2}{3}\,\xi) \right\}^{-1} \left\{ \frac{16}{21}\,\alpha^2\, N^{-2}\,\xi + 8\,\alpha\,\nu\, N^{-1} \right\}.$$

Therefore,

$$C_{22} = (8)^{\frac{1}{4}}\, N^{-\frac{1}{4}}\, (1 - \tfrac{2}{3}\,\xi)^{-\frac{1}{4}} \left\{ \frac{2}{21}\, N^{-1}\,\xi + Pr \right\}^{\frac{1}{4}} \left\{ \frac{g\,\beta\,(T_w - T_\infty)}{\nu^2} \right\}^{-\frac{1}{4}} (Pr)^{-\frac{1}{2}} \tag{3.2.1.37}$$

Substituting Equation (3.2.1.37) into Equation (II.3.2), we get

$$C_{21} = 8\,\alpha\,(8)^{-\frac{1}{2}}\, N^{\frac{1}{2}}\, (1 - \tfrac{2}{3}\,\xi)^{\frac{1}{2}} \left\{ \frac{2}{21}\, N^{-1}\,\xi + Pr \right\}^{-\frac{1}{2}} \left\{ \frac{g\,\beta\,(T_w - T_\infty)}{\nu^2} \right\}^{\frac{1}{2}} Pr\, N^{-1}\,\xi\, .$$

Rearranging the above Equation yields

$$C_{21} = (8)^{\frac{1}{2}} N^{-\frac{1}{2}} \xi \left\{ 1 - \frac{2}{3} \xi \right\}^{\frac{1}{2}} \left\{ \frac{2}{21} \xi N^{-1} + Pr \right\}^{-\frac{1}{2}} \left\{ \frac{g \beta (T_w - T_\infty)}{\nu^2} \right\}^{\frac{1}{2}} \nu \quad . \tag{3.2.1.36}$$

APPENDIX III

III.1. Derivation of Equation (3.2.2.3)

$$\int_0^{\delta_h} (T - T_\infty)\, dy = \int_0^{\delta_t} (T - T_\infty)\, dy + \int_{\delta_t}^{\delta_h} (T - T_\infty)\, dy \ ,$$

However, as the temperature in the region beyond the thermal boundary-layer is equal to the ambient temperature, T_∞ , the second term of the right-hand side in the above equation vanishes.

Therefore,

$$\int_0^{\delta_h} (T - T_\infty)\, dy = \int_0^{\delta_t} (T - T_\infty)\, dy \ . \qquad \text{(III.1.1)}$$

Substituting Equation (3.2.2.2) into Equation (III.1.1) leads to

$$\int_0^{\delta_h} (T - T_\infty)\, dy = \int_0^{\delta_t} \frac{q\,\delta_t}{2k} \left(1 - \frac{y}{\delta_t}\right)^2 dy = \frac{1}{6} \left(\frac{q}{k}\right) \cdot \delta_t^{\,2} \ . \qquad \text{(III.1.2)}$$

Substituting Equations (I.1.1), (I.1.2) and (III.1.2) into Equation (3.1.1) and rearranging, we get

$$\frac{1}{105} \frac{d}{dx} \left(\delta_h \cdot u_x^{\,2} \right) = \frac{1}{6}\, g\beta \left(\frac{q}{k}\right) \zeta^2 \, \delta_h^{\,2} - \nu \frac{u_x}{\delta_h} \ . \qquad (3.2.2.3)$$

III.2. Derivation of Equation (3.2.2.4)

From the boundary condition at the wall we have

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = - \frac{q}{k} \ . \qquad \text{(III.2.1)}$$

Making use of Equations (3.2.2.1) and (3.2.2.2) in evaluating the terms of Equation (3.1.2), we can write

$$\int_0^{\delta_t} (T - T_\infty)\, u\, dy = \int_0^{\delta_t} u_x \left(\frac{y}{\delta_h}\right)\left(1 - \frac{y}{\delta_h}\right)^2 \left(\frac{q\,\delta_t}{2k}\right)\left(1 - \frac{y}{\delta_t}\right)^2 dy$$

$$= u_x \left(\frac{q\,\delta_t}{2k}\right) \int_0^{\delta_t} \left(\frac{y}{\delta_h}\right)\left(1 - \frac{y}{\delta_h}\right)^2 \left(1 - \frac{y}{\delta_t}\right)^2 dy$$

$$= u_x \left(\frac{q\,\delta_t}{2k}\right) \int_0^{\delta_t} \left(\frac{y}{\delta_t}\,\zeta\right)\left(1 - \frac{y}{\delta_t}\,\zeta\right)^2 \left(1 - \frac{y}{\delta_t}\right)^2 dy$$

where $$\zeta = \frac{\delta_t}{\delta_h}\,.$$

Therefore,

$$\int_0^{\delta_t} (T - T_\infty)\, u\, dy = u_x \left(\frac{q\,\delta_t}{2k}\right) \int_0^{\delta_t} \left(\frac{y}{\delta_t}\,\zeta\right)\left\{1 - (1 + \zeta)\,\frac{y}{\delta_t} + \zeta\left(\frac{y}{\delta_t}\right)^2\right\}^2 dy$$

$$= u_x \left(\frac{q}{2k}\right) \delta_t^{\,2}\, \zeta \left\{\frac{1}{12} - \frac{1}{15}\,\zeta + \frac{1}{60}\,\zeta^2\right\}.$$

Thus,

$$\int_0^{\delta_t} (T - T_\infty)\, u\, dy = \frac{1}{6}\left(\frac{q}{k}\right) u_x\, \delta_t^{\,2}\, \zeta\, M \qquad \text{(III.2.2)}$$

where $$M = \frac{1}{4} - \frac{1}{5}\,\zeta + \frac{1}{20}\,\zeta^2\,.$$

Substituting Equations (III.2.1) and (III.2.2) into Equation (3.1.2) leads to

$$\frac{d}{dx}\left\{\frac{1}{6}\left(\frac{q}{k}\right) u_x \delta_t^{\,2} \zeta M\right\} = \frac{\alpha q}{k} ,$$

but, since q is constant along the plate,

$$\frac{1}{6} \frac{d}{dx}(M u_x \cdot \delta_t^{\,2} \zeta) = \alpha . \qquad (3.2.2.4)$$

III.3. Derivation of Equations (3.2.2.11) and (3.2.2.12)

From Equation (3.2.2.10), we have

$$C_{31} = 6 \alpha C_{32}^{\,-2} M^{-1} \zeta^{-3} . \qquad (III.3.1)$$

Substituting Equation (III.3.1) into Equation (3.2.2.9) leads to

$$36 \alpha^2 C_{32}^{\,-3} M^{-2} \zeta^{-6} = 75\left\{\frac{1}{6} g\beta \left(\frac{q}{k}\right) C_{32}^{\,2} \zeta^2 - 6 \alpha C_{32}^{\,-3} M^{-1} \zeta^{-3} \nu\right\}$$

or

$$36 \alpha^2 M^{-2} \zeta^{-6} = 75\left\{\frac{1}{6} g\beta \left(\frac{q}{k}\right) C_{32}^{\,5} \zeta^2 - 6 \alpha M^{-1} \zeta^{-3} \nu\right\} .$$

From this result,

$$C_{32} = 36^{\frac{1}{5}} M^{-\frac{1}{5}} \zeta^{-1}\left\{\frac{2}{25} \zeta^{-3} M^{-1} + Pr\right\}^{\frac{1}{5}} \left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{-\frac{1}{5}} (Pr)^{-\frac{2}{5}} . \qquad (3.2.2.12)$$

Substituting Equation (3.2.2.12) into Equation (III.3.1) yields

$$C_{31} = 6 \alpha (36)^{-\frac{2}{5}} M^{\frac{2}{5}} \zeta^2 \left\{\frac{2}{25} \zeta^{-3} M^{-1} + Pr\right\}^{-\frac{2}{5}} \left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{\frac{2}{5}} (Pr)^{\frac{4}{5}} M^{-1} \zeta^{-3}$$

$$= 36^{\left(\frac{1}{2}-\frac{2}{5}\right)} \alpha M^{\left(\frac{2}{5}-1\right)} \zeta^{2-3} \left\{\frac{2}{25} \zeta^{-3} M^{-1} + Pr\right\}^{-\frac{2}{5}} \left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{\frac{2}{5}} (Pr)^{\frac{4}{5}} .$$

Therefore,

$$C_{31} = 36^{\frac{1}{10}} M^{-\frac{3}{5}} \zeta^{-1} \nu \left\{ \frac{2}{25} \zeta^{-3} M^{-1} + Pr \right\}^{-\frac{2}{5}} \left\{ \frac{g\beta}{\nu^2} \left(\frac{q}{k}\right) \right\}^{\frac{2}{5}} (Pr)^{-\frac{1}{5}} .$$

(3.2.2.11)

III.4. Derivation of Equation (3.2.2.17)

Substituting Equation (3.2.2.15) into Equation (3.2.2.16) leads to

$$T_w - T_\infty = \left(\frac{q}{2k}\right) 36^{\frac{1}{5}} M^{-\frac{1}{5}} \left\{ \frac{2}{25} \zeta^{-3} M^{-1} + Pr \right\}^{\frac{1}{5}} Gr_x^{*\,-\frac{1}{5}} Pr^{-\frac{1}{5}} x$$

$$= \left(\frac{36}{25}\right)^{\frac{1}{5}} \left(\frac{qx}{k}\right) \left\{ \frac{2}{25} \zeta^{-3} M^{-1} + Pr \right\}^{\frac{1}{5}} \left\{ Pr^2 \, Gr_x^{*} \, M \right\}^{-\frac{1}{5}} .$$

Thus,

$$T_w - T_\infty = \left(\frac{9}{8}\right)^{\frac{1}{5}} \left(\frac{qx}{k}\right) \left\{ \frac{0.8\ (10M)^{-1} \zeta^{-3} + Pr}{Pr^2 \, Gr_x^{*} \, M} \right\}^{\frac{1}{5}} . \qquad (3.2.2.17)$$

APPENDIX IV

IV.1. Derivation of Equation (3.2.2.25)

Substituting Equation (3.2.2.2) into the integrand of the right-hand side of Equation (3.2.2.23) leads to

$$\int_0^{\delta_h} (T - T_s)\, dy = \int_0^{\delta_h} \left\{ (T - T_\infty) - (T_s - T_\infty) \right\} dy$$

$$= \frac{q\,\delta_t}{2k} \int_0^{\delta_h} \left\{ (1 - \frac{y}{\delta_t})^2 - (1 - \frac{\delta_h}{\delta_t})^2 \right\} dy$$

$$= \frac{q\,\delta_t}{2k} \int_0^{\delta_h} \left\{ (1 - \xi \frac{y}{\delta_h})^2 - (1 - \xi)^2 \right\} dy$$

$$= \frac{q\,\delta_t}{2k} \int_0^{\delta_h} \left\{ 1 - 2\xi (\frac{y}{\delta_h}) + \xi^2 (\frac{y}{\delta_h})^2 - 1 + 2\xi - \xi^2 \right\} dy$$

$$= \frac{q\,\delta_h^2}{2k} (1 - \frac{2}{3}\xi) \quad . \qquad \text{(IV.1.1)}$$

Substituting Equations (I.1.1), (I.1.2) and (III.1.1) into Equation (3.2.2.23) yields

$$\frac{1}{105} \frac{d}{dx} (\delta_h^2 \cdot u_x^2) = \frac{1}{2} g \beta (\frac{q}{k})\, \delta_h^2 (1 - \frac{2}{3}\xi) - \nu \frac{u_x}{\delta_h} \,.$$

$$\text{(3.2.2.25)}$$

IV.2. Derivation of Equation (3.2.2.26)

Substituting Equations (3.2.2.1) and (3.2.2.2) for the integrand of Equation (3.2.2.24), we get

IV.3. Derivation of Equations (3.2.2.33) and (3.2.2.34)

From Equation (3.2.2.32), we get

$$C_{41} = 6\alpha\ C_{42}^{-2}\ N^{-1}\ \xi\ . \tag{IV.3.1}$$

Substituting Equation (IV.3.1) into Equation (3.2.2.31) yields

$$36\ \alpha^2\ C_{42}^{-3}\ N^{-2}\ \xi^2 = 75\left\{\frac{1}{2}\ g\beta\left(\frac{q}{k}\right) C_{42}^{\ 2}\ \left(1 - \frac{2}{3}\xi\right) - 6\alpha\ C_{42}^{-3}\ N^{-1}\xi\ \nu\right\}$$

$$36\ \alpha^2\ N^{-2}\xi^2 = 75\left\{\frac{1}{2}\ g\beta\left(\frac{q}{k}\right) C_{42}^{\ 5}\ \left(1 - \frac{2}{3}\ \xi\right) - 6\alpha\xi N^{-1}\nu\right\}$$

$$\frac{1}{2}\ g\beta\left(\frac{q}{k}\right) C_{42}^{\ 5}\ \left(1 - \frac{2}{3}\ \xi\right) = \frac{36}{75}\alpha^2\ N^{-2}\ \xi^2 + 6\ \alpha\ N^{-1}\ \xi\ \nu$$

$$C_{42}^{\ 5} = 12\left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{-1}\left\{1 - \frac{2}{3}\xi\right\}^{-1}\left\{\frac{2}{25}\alpha^2\ N^{-2}\ \xi^2 + \alpha\ N^{-1}\ \xi\ \nu\right\}\ \nu^{-2}$$

$$= 12\left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{-1}\left\{1 - \frac{2}{3}\xi\right\}^{-1}\left\{\frac{2}{25}N^{-1}\xi + Pr\right\} Pr^{-2}\ N^{-1}\ \xi$$

$$C_{42} = 12^{\frac{1}{5}}\ N^{-\frac{1}{5}}\ \xi^{\frac{1}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{-\frac{1}{5}}\left\{\frac{2}{25}\ \xi N^{-1} + Pr\right\}^{\frac{1}{5}}\left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{-\frac{1}{5}} Pr^{-\frac{2}{5}}$$

$$\tag{3.2.2.34}$$

Substituting Equation (3.2.2.32) into Equation (IV.3.1) leads to

$$C_{41} = 6\alpha N^{-1}\xi\,(12)^{-\frac{2}{5}}\ N^{\frac{2}{5}}\ \xi^{-\frac{2}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{\frac{2}{5}}\left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{-\frac{2}{5}}\left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{\frac{2}{5}} Pr^{\frac{4}{5}}$$

or

$$C_{41} = 54^{\frac{1}{5}}\ N^{-\frac{3}{5}}\xi^{\frac{3}{5}}\left\{1 - \frac{2}{3}\xi\right\}^{\frac{2}{5}}\left\{\frac{2}{25}\ \xi\ N^{-1} + Pr\right\}^{-\frac{2}{5}}\left\{\frac{g\beta}{\nu^2}\left(\frac{q}{k}\right)\right\}^{\frac{2}{5}} Pr^{-\frac{1}{5}}\ \nu\ .$$

$$\tag{3.2.2.33}$$

IV.4. Derivation of Equation (3.2.2.39)

Substituting Equation (3.2.2.37) into Equation (3.2.2.28) leads to

$$T_w - T_\infty = \left(\frac{qx}{2k}\right) 12^{\frac{1}{5}} N^{-\frac{1}{5}} \xi^{-\frac{4}{5}} \left\{1 - \frac{2}{3}\xi\right\}^{-\frac{1}{5}} \left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{\frac{1}{5}} Gr_x^{*\,-\frac{1}{5}} Pr^{-\frac{2}{5}}$$

$$= \left(\frac{3}{8}\right)^{\frac{1}{5}} \left(\frac{qx}{k}\right) \xi^{-\frac{4}{5}} \left\{1 - \frac{2}{3}\xi\right\}^{-\frac{1}{5}} \left\{\frac{2}{25}\xi N^{-1} + Pr\right\}^{\frac{1}{5}} Gr_x^{*\,-\frac{1}{5}} Pr^{-\frac{2}{5}} N^{-\frac{1}{5}} .$$

Rearranging the above Equation yields

$$T_w - T_\infty = \left(\frac{3}{8}\right)^{\frac{1}{5}} \left(\frac{qx}{k}\right) \left\{\frac{0.8\ (10N)^{-1}\xi + Pr}{Pr^2\ Gr_x^{*}\ N \cdot \xi\ (1 - \frac{2}{3}\xi)}\right\}^{\frac{1}{5}} . \qquad (3.2.2.39)$$

LAMINAR FREE-CONVECTION ON A VERTICAL FLAT PLATE WITH UNIFORM SURFACE TEMPERATURE OR UNIFORM SURFACE HEAT FLUX

by

JEROME CHIEH-JAN LEE

B. S., National Taiwan University, Taipei, Taiwan, 1950

AN ABSTRACT OF A MASTER'S REPORT

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE UNIVERSITY
Manhattan, Kansas

1967

The problem of laminar free-convection on a vertical plate with uniform wall temperature or uniform heat flux at the wall was analyzed, using the Karman-Pohlhausen method. The following four cases were investigated:

1. Vertical plate at constant wall temperature, for fluids of $Pr > 1$
2. Vertical plate with uniform wall heat flux, for fluids of $Pr > 1$
3. Vertical plate at constant wall temperature, for fluids of $Pr < 1$
4. Vertical plate with uniform wall heat flux, for fluids of $Pr < 1$

Unlike previous investigations, the thickness of the velocity boundary-layer and the thickness of the thermal boundary-layer were assumed different in this analysis. Also, the ratio of the two thicknesses was assumed to be a function of Pr only. Expressions for Nu_x , for the four cases investigated were derived. The derived expressions required a knowledge of the ratio of thicknesses of the thermal boundary-layer to the velocity boundary-layer as a function of Pr . This ratio was obtained by matching the prediction of $Nu_x/(Gr_x/4)^{\frac{1}{4}}$ or $Nu_x/(Gr_x^*)^{1/5}$ of the present analysis with the prediction of exact solutions of earlier investigations. The results of the present analysis were compared with results of other approximate investigations.